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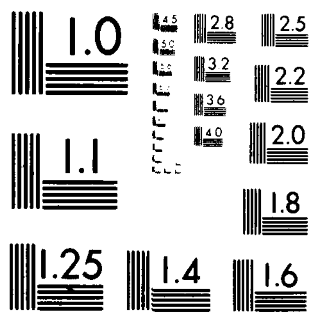

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TECHNICAL REPORT

SCHEME INDEPENDENT STABILITY CRITERIA FOR DIFFERENCE APPROXIMATIONS TO HYPERBOLIC INITIAL BOUNDARY VALUE SYSTEMS

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20. Abstract

We consider finite difference approximations to the mixed initial boundary value problems of the hyperbolic system $v_t = a u_x$ in the quarter plan $x \geq 0$, $t \geq 0$. The finite difference approximations may be either dissipative or non-dissipative. Easily checkable sufficient stability criteria are obtained entirely in terms of the boundary conditions for the out flow variables.

It is shown how to extend the results for the two dimensioanl case when the boundary conditions in the y direction allow Fourier transformations. Examples are given.

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TABLE OF CONTENTS

INTRODUCTION

Finite difference methods are today one of the important tools for approximating the solutions of time-dependent problems governed by systems of partial differential, equations. This is ture for example for a wide spectrum of pure initial value and mixed initial-boundary value problems in the field of fluid-dynamics. The widespread use of finite difference schemes to solve such problems has increased rapidly since the early 1950's, in response to the increased capabilities of the electronic computers which execute the vast amount of calculations needed in the applications.

Since that time there have also been extensive developments in the analysis of finite difference schemes. A prime example of this has been the maturation of the mathematical theory needed to handle numerical approximations to linear systems of initial value problems. The concept of stability of a difference scheme, expressing a continuous dependence of the scheme-solution on its initial values, plays a major role in the above mentioned theory. The centrality of this concept follows from the Lax equivalence theorem (see for example [13, Chapter 3] which assures the convergence of a numerical computation carried out by a *stable* finite difference scheme consistent with a well-posed initial value problem.

Besides the pure initial value problems mentioned above,we are most interested in approximating the solution of mixed initial-boundary value problems, where the numerical approximation must include a special boundary treatment to fulfil the boundary conditions imposed on the problem. Furthermore, since practically all the various schemes are solved in a finite domain of the grid, it follows that numerical boundary conditions must be added also in the case of pure initial value problems so that the solution can be uniquely determined. It follows that in all cases (pure initial value and mixed initial-boundary value problems), the overall approximation is composed of a basic scheme applied at inner grid points and a (different) additional algorithm which is applied locally at the boundary. The additional boundary treatment which determines uniquely the scheme-solution is sometimes an artificial one and does not necessarily reflect the boundary conditions (if any) of the original differential system, so it may cause an instability. Indeed it is known (see for example [13, Chapter 6] [11, Chapter 17]) that even if the basic scheme is stable, a careless numerical boundary treatment may render the total computation unstable.

These considerations lead us into the area of the stability analysis of approximations to hyperbolic initial-boundary value problems. One of the most important contributions in that area which will serve us as a general reference on the subject, is the 1972 paper by Gustafsson, Kreiss and Sundström [6] which is a generalization of an earlier paper by Kreiss (1968) [8]. The analysis in the 1972 paper rests on a new stability definition (Definitions 3.2 and 3.3 in [6]) which like the stability definition for approximations to pure initial value problems, is obtained by a discretization of a corresponding well-posedness condition of the original differential equation (see for example [9]). This new stability condition reflects the influence that the *boundary values* have on the numerical solution, and as in the case of pure initial value problems, it serves as a sufficient condition for the convergence of a (compatible) consistent approximation [5]. The main result of Gustafsson et. al.[6] provides an algebraic criterion which enables one to determine whether a given approximation, consisting of a basic scheme together with corresponding boundary conditions, is stable or not. Roughly speaking, according to this criterion we have stability if (and only if) no linear combination of powers of roots satisfying a characteristic equation which is determined by the basic scheme, may serve as a non-trivial solution of some characteristic boundary constraints. Thus, in order to assure stability by applying the above stability criteria, one must first study the behavior of roots of the corresponding characteristic equation. This characteristic equation is a polynomial equation with $N \times N$ matrix coefficients, N denoting the order of the original approximated system, and whose degree depends on the number of spatial mesh points that the basic scheme rests on. Studying the behavior of the roots of such an equation as part of applying the above stability criteria for *general* difference schemes, is a complicated task which cannot always be carried out fully analytically. Therefore, examples given in the literature of verifying stability for initial-boundary approximations, are in most cases restricted to *specific scalar 3-point schemes*.

This fact motivates us to look for simpler sufficient stability tests. Scheme-independent stability tests which are exclusively dependent on the boundary conditions, are particularly useful for such purpose. Tests of this sort have two main advantages: first, their being independent of the basic scheme relieves us of the technical difficulties associated with the computation of roots of the characteristic equation and hence the procedure of checking stability becomes much shorter; and secondly, the acquired stability is not restricted to a specific approximation but instead is valid for a family of basic schemes which are characterized by some general property.

The search for such scheme-independent stability criteria for difference approximations to hyperbolic initial-boundary value systems is the main subject of the dissertation.

As a model problem for the general linear case we consider the hyperbolic system $u_t = Au_x + F$ which together with appropriate initial and boundary conditions is well-posed in the quarter plane $x \geq 0$, $t \geq 0$. In prescribing these appropriate boundary conditions it turns out that one must distinguish between inflow and outflow (characteristic) unknowns, where the inflow boundary values have to be determined by reflection of the outflow ones. Based on this distinction we first introduce in Chapter 1 a general method of numerical boundary treatment of arbitrary degree of accuracy, such that the entire vector approximation is stable if and only if the *scalar* components of its outflow part are; thus reducing the stability question to that of a *scalar* (outflow) problem. Therefore from that point on our discussion concentrates on the general scalar approximation as it is represented in the second part of Chapter 1.

In Chapter 2 we begin the stability study, drawing on the stability theory of Gustafsson et. al.[6] which we briefly survey in the first section of that chapter. The main stability criterion in that theory is given in terms of eigenvalues and generalized eigenvalues of the problem. Then, upon reintroducing these concepts in a less formal manner and operating under the four basic assumptions corresponding to those which were made in [6], we may apply the above criterion [6, Theorem 5.1] which states that a given initial-boundary approximation is stable if and only if it has neither eigenvalues nor generalized eigenvalues outside the unit disc.

In the second section of Chapter 2 we follow the analysis in [6] which leads to the formulation of the main stability criterion as a corresponding determinant condition. Then, by using in the above analysis, a suitable representation of the general form of (generalized) eigenvalues of the problem, we derive an *explicit* interpretation of the determinantal stability criterion mentioned above. This result, which seems to be of independent interest, is essential for the general stability analysis which is carried out afterwards and is needed to obtain the stability criteria of the desired type.

Chapter 3 -- the main one in this work -- discusses scheme-independent stability criteria. The boundary conditions considered are of translatory type, i.e., determined at all points in the boundary domain by the same procedure. We first show in Section 3.1 that when dealing with such boundary condition, the determinental stability criterion obtained in Chapter 2 is equivalent to a corresponding *scalar* condition. This scalar condition plays the central role in proving the scheme-independent results at which we are aiming.

In the remainder of Chapter 3 we state our main results, namely, sufficient scheme-independent stability criteria. These results are obtained upon making two quite non-restrictive assumptions complementing the first four already made in Chapter 2. It is shown that these new additional assumptions are necessary for our scheme-independent results to be valid, and simple scheme-independent tests verifying whether a given problem meets these assumptions, are provided. We study the cases of both one-level and multi-level boundary treatments. In the (somewhat simpler) first case, the well-known result (see for example [3], [7]) stating that two-level stable dissipative schemes together with (one-leveled) extrapolated outflow boundary values remain as conditionally stable, follows easily. We show however, that this widely used result is no longer valid when dealing with *multi-level* dissipative schemes involving more than two time-steps, unless further restrictions are made on them. In studying the wide class of multi-level boundary treatments, we will employ the tools of dissipativity and the von Neumann condition usually used only in connection with the basic scheme. The stability criteria in the multi-level case are given in terms of these concepts which are *well-understood* from the theory of pure initial value problems and whose validity can be *easily checked.* We prove that *an arbitrary stable dissipative scheme when complemented by outflow boundary conditions satisfying the von Neumann condition, remains stable.* We also show that *if the outflow boundary conditions are dissipative, then the entire approximation is stable independently of the interior scheme* (be it dissipative or a non-dissipative one).

Finally, in Chapter 4, we utilize the above scheme-independent stability criteria to verify the stability of various (outflow) translatory boundary conditions. The examples considered indicate that an *arbitrary* stable dissipative scheme whose outflow boundary values are translatorily computed by oblique extrapolation, by the Box-Scheme or by the stable weighted Euler scheme, constitutes a stable approximation. We also study boundary conditions which are generated by the (right-sided) implicit and stable explicit Euler schemes. Both boundary treatments are found to be *unconditionally stable* in the sense that when augmenting *arbitrary* stable basic schemes, they always maintain stability. We close Chapter

4 by considering approximations to the two space dimensional problem $u_t = au_x + bu_y$ in the quarter space $x \geq 0$, $t \geq 0$, $-\infty < y < \infty$. The stability analysis in that case is based on Fourier transforming with respect to y (with dual variable η), thus obtaining a *one space dimensional* problem of the type analyzed in previous chapters with η-dependent coefficients.

1. THE DIFFERENCE APPROXIMATION TO THE HYPERBOLIC SYSTEM

1.1. The reduction to the scalar problem

Consider a first order hyperbolic system of partial differential equations

$$\partial u(x,t)/\partial t = A\partial u(x,t)/\partial x + F(x,t) \tag{1.1a}$$

in the quarter-plane $x \geq 0$, $t \geq 0$, with initial conditions

$$u(x,0) = f(x) \;,\quad 0 \leq x < \infty \;. \tag{1.1b}$$

Here, $u(x,t) \equiv (u^{(1)}(x,t),\ldots,u^{(N)}(x,t))'$ *is the transposed vector of unknowns*, A is $N \times N$ non-singular constant coefficient matrix and $F(x,t) \equiv (F^{(1)}(x,t),\ldots,F^{(N)}(x,t))'$, $f(x) \equiv (f^{(1)}(x),\ldots,f^{(N)}(x))'$, are N-dimensional vector functions.

The hyperbolicity of the system (1.1a) implies that A can be diagonalized by a similarity transformation, hence we may assume without restriction that A is already given in its diagonal form

$$A = \begin{pmatrix} A^- & \\ & A^+ \end{pmatrix};\; A^- = \begin{pmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_L \end{pmatrix} < 0,\; A^+ = \begin{pmatrix} a_{L+1} & & 0 \\ & \ddots & \\ 0 & & a_N \end{pmatrix} > 0 \;. \tag{1.1c}$$

We are interested in the uniqueness of the solution for the system (1.1a). For that reason, let us consider the partition

$$u^- \equiv (u^{(1)},\dots,u^{(L)})' \ , \ u^+ \equiv (u^{(L+1)},\dots,u^{(N)})'$$

corresponding to that of A. Since the characteristic lines associated with the $N-L$ components of $u^+(x,t)$ go outside from the region $x,t \geq 0$, it follows that $u^+(x,t)$ which is carried by these characteristics, is uniquely determined by the initial values $f(x)$ in the whole quarter-plane $x,t \geq 0$. Because of the direction of its characteristics, $u^+(x,t)$ is considered as the *outgoing* part of the solution $u(x,t)$.

The L characteristic lines associated with $u^-(x,t)$ have a positive slope and hence go into the quarter-plane $x,t \geq 0$. Therefore, in order that $u^-(x,t)$, which is considered as the *ingoing* part of the solution $u(x,t)$, be uniquely determined in the quarter-plane, it is necessary to specify its values on the boundary line $x = 0$. Thus, for the solution of (1.1a) to be uniquely determined, we prescribe boundary conditions of the general form

$$u^-(0,t) = Su^+(0,t) + g(t) \ , \ t \geq 0, \tag{1.1d}$$

which determine the missing ingoing boundary values by reflection of the outgoing ones. Here, S is an $L\times(N-L)$ constant matrix and $g(t) \equiv (g^{(1)}(t),\dots,g^{(L)}(t))'$ is an L-dimensional vector function.

To solve the initial-boundary value problem (1.1) by a difference approximation we introduce a mesh-size $h \equiv \Delta x > 0$, $\Delta t > 0$ such that $\lambda \equiv \Delta t/\Delta x = \text{constant}$. Using the standard notation $v_\nu(t) \equiv v(\nu h,t)$, we approximate (1.1a) by a consistent, two-sided multi-level scheme

$$Q_{-1}v_\nu(t+\Delta t) = \sum_{\sigma=0}^{s} Q_\sigma v_\nu(t-\sigma\Delta t) + \Delta t\cdot F_\nu(t), \quad \nu = 1,2,\ldots, \tag{1.2a}$$

with initial values

$$v_\nu(\sigma\Delta t) = f_\nu(\sigma\Delta t), \quad \nu \geq -r+1, \ \sigma = 0,1,\ldots,s. \tag{1.2b}$$

Here

$$Q_\sigma = \sum_{j=-r}^{p} A_{j\sigma}E^j \ , \quad Ev_\nu = v_{\nu+1} \ ,$$

are difference operators with matrix coefficients depending on A and on λ.

In order to determine uniquely the solution of (1.2), we must specify at each time step the r boundary values $v_\mu(t)$, $\mu = 0,-1,\ldots,-r+1$. For the approximated outgoing unknowns $v_\nu^+(t) \equiv (v_\nu^{(L+1)}(t),\ldots,v_\nu^{(N)}(t))'$ we do it by boundary conditions of the form

$$\begin{gathered} S_{-1}^{(\mu)}v_\mu^+(t+\Delta t) = \sum_{\sigma=0}^{\tau} S_\sigma^{(\mu)}v_\mu^+(t-\sigma\Delta t) + \Delta t\cdot F_\mu(t) \ , \quad \mu = 0,-1,\ldots,-r+1, \\ S_\sigma^{(\mu)} = \sum_{j=0}^{q} C_{j\sigma}^{(\mu)}E^j \ , \quad \sigma = -1,0,1,\ldots,\tau \ , \quad \tau \geq -1, \end{gathered} \tag{1.3}$$

where $C_{j\sigma}^{(\mu)}$ are $(N-L)\times(N-L)$ constant matrices depending on A and on λ. In that way, we maintain the property shared by the analytic system (1.1), which is the unique determination of the outgoing unknowns in the whole quarter-plane *independently of the ingoing ones*.

In (1.3), $\tau + 1$ $(\tau \geq -1)$ denotes the number of time levels needed for the computation of $v_\mu(t+\Delta t)$. The case $\tau = -1$ is considered to be the case of one-level boundary conditions where the first term on the right side of (1.3) is taken to be zero.

For the computation of the boundary values of the approximated ingoing unknowns $v_\nu^-(t) \equiv (v_\nu^{(1)}(L),\ldots,v_\nu^{(L)}(t))'$ we use the analytic boundary condition

(1.4a) $$v_0^-(t) = Sv_0^+(t) + g(t),$$

together with $r-1$ additional conditions of the form

(1.4b) $$v_\mu^-(t) = \sum_{j=-r+1}^{q} D_{\mu j} v_j^+(t) + g_\mu(t) \ , \quad \mu = -1,-2,\ldots,-r+1.$$

Here, $D_{\mu j}$ are $L\times(N-L)$ constant matrices and the $g_\mu(t)$ are L-dimensional bounded vector functions depending on h and on $g(t)$. In other words, as in the analytic case, the computation of the ingoing boundary values is based on reflection of the outgoing ones.

It is well-known that using conditions of the general form (1.3), one can achieve at the boundary, arbitrary degrees of accuracy. We note that this is true also for conditions of the type (1.4b). Indeed, if accuracy of order d is desired we can use the Taylor expansion of a smooth solution for (1.1)

$$u_\mu^-(t) = \sum_{j=0}^{d} \frac{(\mu h)^j}{j!} \frac{\partial^j}{\partial x^j} [u^-(0,t)] + \mathcal{O}(h^{d+1}),$$

and by the differential system (1.1a) and (1.1d) we get for a typical spatial derivative in the above expansion

$$\frac{\partial^j}{\partial x^j}u^-(0,t) = (A^-)^{-j}\frac{\partial^j}{\partial t^j}u^-(0,t) = (A^-)^{-j}[S\frac{\partial^j}{\partial t^j}u^+(0,t) + \frac{d^j}{dt^j}g(t)] =$$

$$= (A^-)^{-j}[S(A^+)^j\frac{\partial^j}{\partial x^j}u^+(0,t) + \frac{d^j}{dt^j}g(t)].$$

Thus, (1.4b) follows upon approximating $\partial^j/\partial x^j u^+(0,t)$ by linear combinations of $u^+_{-r+1}(t),\dots,u^+_q(t)$ of the right accuracy.

The difference approximation is completely defined now by the (basic) scheme (1.2a) together with the boundary conditions (1.3), (1.4) and we raise the question of its overall stability which means, according to Definition 3.3 in [6], that the discrete solution $v_\nu(t)$ could be estimated with the aid of the inhomogeneous terms $F_\nu(t)$, $\nu \geq -r+1$, $g(t)$ and $g_\mu(t)$, $\mu = -1,-2,\dots,-r+1$.

For that purpose, we split the scheme (1.2a) into its inflow and outflow parts

(1.5) $$Q^-_{-1}v^-_\nu(t+\Delta t) = \sum_{\sigma=0}^{s} Q^-_\sigma v^-_\nu(t-\sigma\Delta t) + \Delta t\cdot F^-_\nu(t),\ \nu = 1,2,\dots,$$

(1.6) $$Q^+_{-1}v^+_\nu(t+\Delta t) = \sum_{\sigma=0}^{s} Q^+_\sigma v^+_\nu(t-\sigma\Delta t) + \Delta t\cdot F^+_\nu(t),\ \nu = 1,2,\dots,$$

which are coupled through the boundary conditions (1.4). Here, Q^-_σ, Q^+_σ are difference operators which are given by

$$Q^-_\sigma = \sum_{j=-r}^{p} A^-_{j\sigma}E^j\ ,\ Q^+_\sigma = \sum_{j=-r}^{p} A^+_{j\sigma}E^j\ ;\ A_{j\sigma} \equiv \begin{pmatrix} A^-_{j\sigma} & \\ & A^+_{j\sigma} \end{pmatrix}.$$

Thus, Q_σ^-, Q_σ^+ denote respectively the partition of the difference operators Q_σ into their inflow and outflow parts, according to the dependence of the matrix coefficients $A_{j\sigma}$ on A^- and A^+.

In order to assure the stability of the entire approximation, both of its parts - the outflow part (1.6), (1.3) and the inflow part (1.5), (1.4) - have to be stable. We note that the outflow approximation (1.6), (1.3) is independent of the inflow values, while the inflow approximation (1.5), (1.4) depends on the outflow part only to the extent that the outflow computations provide the inhomogeneous boundary values in (1.4b).

Let us consider first the stability of the self-contained outflow approximation (1.6), (1.3). Since the difference operators Q_σ^+, $S_\sigma^{(\mu)}$ are expressed in terms of $A_{j\sigma}^+$ and $C_{j\sigma}^{(\mu)}$ which in turn depend on the diagonal matrix A^+, it follows that the outflow problem splits into N-L *independent scalar approximations*. Thus, the outflow problem is stable if and only if its N-L scalar component approximations are.

Now, suppose the outflow approximation was found stable, then it remains to determine whether the same holds true for the inflow approximation (1.5), (1.4). Since the outflow values determined by the stable outflow computation are bounded, then the summation $\sum_{j=-r+1}^{q} D_{\mu j} v_j^+(t)$, which appears on the right side of (1.4b) is a bounded term, *independent of the inflow computation*. Thus, the right side of (1.4b) consists of two bounded terms, which are independent of the inflow computation and therefore, for the purpose of determining stability, it may be considered as an arbitrary inhomogeneous term that provides the ingoing boundary values.

Recalling that the difference operators Q_σ^- defining the basic inflow scheme are expressed in terms of $A_{j\sigma}^-$, which in turn depend on the diagonal matrix A^-, it follows that the inflow problem splits into L *independent scalar initial-boundary approximations* the boundary values of which are determined by some inhomogeneous bounded terms. Thus, the inflow problem is stable if and only if its L corresponding scalar components are.

Concerning the stability question of the initial-boundary approximation which is discussed above, it is obviously necessary to require the stability of its basic scheme should it be applied to the pure initial-value problem, $-\infty < \nu < \infty$

Assume that the basic inflow scheme indeed satisfies the above necessary stability requirement. Then, as we shall see later on, it follows that each scalar component of the inflow approximation whose boundary values are determined by an arbitrary bounded term, is unconditionally stable, [7], [10], and we therefore obtain the unconditional stability of the entire inflow part, (1.5), (1.4).

We conclude that the entire approximation is stable if and only if its scalar components are. Furthermore, according to the remark above, it is sufficient to consider only the outflow ones. Thus, in both cases - either the inflow case or particularly the outflow one, it is the scalar approximation the stability of which we have to look for, so hereafter we may restrict our discussion to the scalar approximation, bearing in mind that our forthcoming results go over to the general vector case.

1.2 The scalar approximation and its solvability

Consider the scalar hyperbolic initial-boundary value problem

(1.7a) $\partial u(x,t)/\partial t = a\partial u(x,t)/\partial x$; $a = \text{constant} \neq 0$; $u(x,0) = f(x)$; $x,t \geq 0$.

Whereas the outflow problem, $a > 0$, is well-posed in $L_2(0,\infty)$, the inflow problem, $a < 0$, is not, unless suitable boundary conditions are given at $x = 0$. Therefore, we shall examine (1.7a) together with

- no boundary conditions for the outflow problem, $a > 0$;

(1.7b)

- boundary conditions $u(0,t) = g(t)$, $t \geq 0$ for the inflow problem, $a < 0$.

To approximate (1.7) numerically, we set a time step $\Delta t > 0$ and a mesh width $h \equiv \Delta x > 0$, a grid function $v_\nu(t) \equiv v(\nu h,t)$, $\nu = 0, \pm 1, \pm 2, \ldots$, and a consistent multi-level finite difference scheme

(1.8)
$$Q_{-1}v_\nu(t+\Delta t) = \sum_{\sigma=0}^{s} Q_\sigma v_\nu(t-\sigma\Delta t); \quad \nu = 1,2,\ldots, t \geq s\Delta t$$
$$Q_\sigma = \sum_{j=-r}^{p} a_{j\sigma}E^j \; ; \; \sigma = -1,0,\ldots,s; \; E^j v_\nu = v_{\nu+j}.$$

Here, $r,p > 0$ and s are natural numbers and the $a_{j\sigma}$'s are constants which depend on the coefficient a and the fixed ratio $\lambda \equiv \Delta t/\Delta x = \text{constant}$.

We note that the consistency of scheme (1.8), i.e., its being at least first order accurate, may be characterized by the following two equations:

zero order accuracy requirement

(1.9a)
$$\sum_{j=-r}^{p} a_{j,-1} = \sum_{\sigma=0}^{s}\sum_{j=-r}^{p} a_{j\sigma},$$

and the additional requirement for first order accuracy

(1.9b)
$$\sum_{j=-r}^{p} ja_{j,-1} = \sum_{\sigma=0}^{s}\sum_{j=-r}^{p} ja_{j\sigma} - \lambda a \sum_{\sigma=0}^{s} (\sigma+1) \sum_{j=-r}^{p} a_{j\sigma} .$$

The equalities (1.9a), (1.9b) may be written respectively, in the following compact form

(1.10a)
$$\sum_{j=-r}^{p} a_j(z)|_{z=1} = 0,$$

(1.10b)
$$\sum_{j=-r}^{p} ja_j(z)|_{z=1} = -\lambda a \sum_{j=-r}^{p} a_j'(z)|_{z=1} \; ; \; [\;\;]' \equiv \frac{d[\;\;]}{dz} ,$$

where the scalar functions $a_j(z)$ are defined by

(1.11)
$$a_j(z) = -\sum_{\sigma=0}^{s} z^{-\sigma-1} a_{j,\sigma} + a_{j,-1} \;, \; -r \le j \le p,$$

and following [6], [8], we shall operate under

<u>ASSUMPTION I</u> (Assumption 5.5 in [6]).

$$a_{-r}(z),\; a_p(z) \ne 0 \;,\; |z| \ge 1.$$

It is clear that under Assumption I, the vector coefficients, $(a_{-r,-1}, a_{-r,0}, \dots, a_{-r,s})'$ does not vanish, and since $r > 0$ it follows that in order to assure the uniqueness of the solution of (1.8), we have to supply its discrete values at the boundary points x_μ, $\mu = 0, -1, \dots, -r+1$. These will be defined via boundary conditions of the form

(1.12a)
$$S_{-1}^{(\mu)} v_\mu(t+\Delta t) = \sum_{\sigma=0}^{\tau} S_\sigma^{(\mu)} v_\mu(t-\sigma\Delta t); \; \mu = 0,-1,\dots,-r+1, \; t \geq \tau\Delta t,$$
$$S_\sigma^{(\mu)} = \sum_{j=0}^{q} c_{j\sigma}^{(\sigma)} E^j \; ; \; \sigma = -1,0,\dots,\tau, \; \tau \geq -1.$$

Here, the $c_{j\sigma}^{(\mu)}$'s are constants which depend on a and λ, q is a natural number and $\tau+1$, $\tau \geq -1$, indicates the number of previous time levels which we need in order to compute the boundary values at the next time level, $t+\Delta t$. We note that the one-level boundary conditions namely $\tau = -1$, is a special case of (1.12a) whereupon (1.12a) takes the form

(1.12b)
$$S_{-1}^{(\mu)} v_\mu(t+\Delta t) = 0, \quad \mu = 0,-1,\dots,-r+1, \; t \geq 0.$$

It is clear that the computation of the boundary values via the r boundary relations (1.12), the linear independence of which is assured by taking

(1.13)
$$c_{0,-1}^{(\mu)} \neq 0 \quad , \quad \mu = 0,-1,\dots,-r+1,$$

is done in the specified order, namely, $\mu = 0,-1,\dots,-r+1$.

Now, the basic scheme (1.8) together with the boundary conditions (1.12) completely define the finite difference approximation, whose numerical solution is initiated with the aid of the initial values given by

$$v_\nu(\sigma\Delta t) = f_\nu(\sigma\Delta t) \ , \ \nu \geq -r+1, \quad \sigma = 0,1,\ldots,s. \tag{1.14}$$

Following [6], we define the *solvability* of the difference approximation as the property of being able to *uniquely* obtain *bounded* grid values at $t+\Delta t$ by applying (1.8) and (1.12), thus making use of the discrete values which were already computed at previous time levels.

To ensure solvability, we consider the space $\ell_2(x)$, of all the grid functions $w = \{w_\nu\}_{\nu=-r+1}^{\infty}$ satisfying $\sum_{\nu=-r+1}^{\infty} |w_\nu|^2 < \infty$. Upon defining respectively an inner product and a norm by

$$(v,w)_x = \Delta x \cdot \sum_{\nu=-r+1}^{\infty} v_\nu \bar{w}_\nu \quad , \quad \|w\|_x^2 = (w,w)_x,$$

$\ell_2(x)$ becomes a Hilbert space, a discrete analogue to $L_2(0,\infty)$. Now, denote by w_ν, $\nu \geq -r+1$, the discrete values to be computed at the next time level, $t+\Delta t$ and rewrite the approximation (1.8), (1.12) in the form

$$Q_{-1}w_\nu = \Psi_\nu \quad \nu = 1,2,\ldots; \quad S_{-1}^{(\mu)}w_\mu = \Psi_\mu \quad \mu = 0,-1,\ldots,-r+1, \tag{1.15}$$

where $\Psi = \{\Psi_\nu\}_{\nu=-r+1}^{\infty}$ $\Psi \in \ell_2(x)$, stands for the linear combinations of previously computed values, as given on the right sides of (1.8) and (1.12).

The solvability of the approximation, which will be henceforth assumed throughout this work, is thus cast in the following form.

ASSUMPTION II (Assumption 3.1 in [6]).

There exists a constant $K_0 > 0$, such that for every $\psi \in \ell_2(x)$ there is a unique solution $w \in \ell_2(x)$ for (1.15) with

$$\|w\|_x^2 \leq K_0^2 \cdot \|\psi\|_x^2 .$$

We note that the solvability condition is automatically fulfilled in case that the basic scheme (1.8) is explicit, i.e., $Q_{-1} \equiv \text{constant} \cdot I$. Concerning the solvability in the general implicit case, the following result due to Osher [12] (based on Strang's earlier paper [16]) holds.

LEMMA 1.1.

Let the index r_0, $0 \leq r_0 \leq r$ *be defined by*

$$r_0 = \max\{j \mid a_{-j,-1} \neq 0,\ 0 \leq j \leq r\} \tag{1.16}$$

and let κ_j, $j = 1,2,\ldots,m$ *be the* m *roots counted according to their multiplicities, of*

$$Q_{-1}(\kappa) = \sum_{j=-r_0}^{p} a_{j,-1}\kappa^j = 0, \tag{1.17}$$

which are lying inside the unit disc $0 < |\kappa_j| < 1$.

I. (Theorem I in [12]). The following three conditions are necessary and sufficient for solvability:

(1.18a) $$Q_{-1}(e^{i\xi}) \neq 0 \ , \ |\xi| \leq \pi$$

(1.18b) $$m = r_0$$

(1.18c) the associated problem with (1.15) which consists of the basic scheme

(1.19a) $$Q_{-1}w_\nu = \sum_{j=-r_0}^{p} a_{j,-1}w_{\nu+j} = 0, \quad \nu = 1,2,\ldots,$$

together with the inhomogeneous boundary conditions

(1.19b) $$S_{-1}^{(\mu)}w_\mu = \Psi_\mu \ , \ \mu = 0,-1,\ldots,-r+1,$$

has a unique solution in $\ell_2(x)$.

II. Let the basic scheme (1.19a) be a right-sided one, i.e., $r_0 = 0$. Then

(1.20) $$Q_{-1}(\kappa) \neq 0 \ , \ 0 < |\kappa| \leq 1$$

is a sufficient condition for solvability.

In particular solvability follows for explicit basic schemes, where we have $r_0 = 0$, $Q_{-1}(\kappa) \equiv$ constant.

PROOF. By (1.18a) we may apply the Argument Principle for $Q_{-1}(\kappa)$ on the unit circle $|\kappa| = 1$, obtaining that (1.18b) is valid if and only if $(2\pi)^{-1}\cdot\int_{-\pi}^{\pi} d[\arg Q_{-1}(e^{i\xi})] = 0$. Hence, conditions (1.18) are exactly those of Osher ((d) (e) and (g) in [12]) and by Theorem I in [12] they are equivalent to solvability.

To prove the second part of the lemma we first note that (1.20) implies (1.18a). Also (1.20) implies that $Q_{-1}(\kappa) = 0$ has no solutions inside the unit disc, i.e., $m = 0$, and since $r_0 = 0$ we have (1.18b) as well. Finally since by (1.20) $Q_{-1}(\kappa) = 0$ has no solutions in the closed unit disc, it follows that the most general solution of (1.19a) in $\ell_2(x)$ must vanish, i.e., $w_\nu = 0$, $\nu = 1,2,\ldots$. In addition the boundary values w_μ, $\mu = 0,-1,\ldots,-r+1$, can be uniquely computed by applying (1.19b) in the successive order $\mu = 0,-1,\ldots,-r+1$, so we get (1.18c). Having (1.18a), (1.18b), (1.18c), part I of the lemma completes the proof.

2. STABILITY ANALYSIS

2.1. The stability definition and Gustafsson's et. al. Main Theorem [6]

In a similar way to the above definition of the space $\ell_2(x)$, we introduce the discrete spaces $\ell_2(t)$ and $\ell_2(x,t)$, which become Hilbert spaces upon defining respectively an inner product and a norm by

$$(v,w)_t = \Delta t \cdot \sum_{\sigma=0}^{\infty} v(\sigma\Delta t)\bar{w}(\sigma\Delta t) \; ; \quad \|w\|_t^2 = (w,w)_t,$$

$$(v,w)_{x,t} = \Delta t \cdot \Delta x \cdot \sum_{\sigma=0}^{\infty} \sum_{\nu=-r+1}^{\infty} v_\nu(\sigma\Delta t)\bar{w}_\nu(\sigma\Delta t); \quad \|w\|_{x,t}^2 = (w,w)_{x,t}.$$

Now, let us write the difference approximation (1.8), (1.12), in the operational form

$$(2.1) \qquad G_{-1}v(t+\Delta t) = \sum_{\sigma=0}^{\max(\tau,s)} G_\sigma v(t-\sigma\Delta t); \; v(t-\sigma\Delta t) \in \ell_2(x),$$

where $G_\sigma : \ell_2(x) \longrightarrow \ell_2(x)$ are linear bounded operators determined by the basic scheme (1.8) together with the boundary conditions (1.12). Here, the solvability assumption II, is expressed by the fact that G_{-1} has a bounded inverse in the whole of $\ell_2(x)$.

DEFINITION 2.1 (Definition 3.3 in [6]).

Consider the inhomogeneous approximation associated with (2.1)

$$(2.2) \qquad G_{-1}w(t+\Delta t) - \sum_{\sigma=0}^{\max(\tau,s)} G_\sigma w(t-\sigma\Delta t) = \Delta t \cdot F \quad F \equiv \{F_\nu\}_{\nu=-r+1}^{\infty} \in \ell_2(x),$$

together with vanishing initial values $f_\nu(\sigma\Delta t) = 0$. The approximation (1.8), (1.12) is said to be *stable* if there exist constants $K_0 > 0$ and $\alpha_0 \geq 0$, such that for every $F \in \ell_2(x)$ and every α, $\alpha \geq \alpha_0$, the solution $w = \{w_\nu\}_{\nu=-r+1}^{\infty}$, of (2.2) satisfies the estimate

$$(2.3) \qquad \left(\frac{\alpha-\alpha_0}{\alpha\cdot\Delta t+1}\right)\cdot\sum_{\mu=-r+1}^{0} \|e^{-\alpha t}w_\mu\|_t^2 + \left(\frac{\alpha-\alpha_0}{\alpha\cdot\Delta t+1}\right)^2\cdot\|e^{-\alpha t}w\|_{x,t}^2 \leq$$

$$\leq K_0^2\cdot\left\{\left(\frac{\alpha-\alpha_0}{\alpha\cdot\Delta t+1}\right)\cdot\sum_{\mu=-r+1}^{0}\|\Delta t\cdot e^{-\alpha(t+\Delta t)}F_\mu\|_t^2 + \|e^{-\alpha(t+\Delta t)}F\|_{x,t}^2\right\} .$$

It is of course understood that before turning to investigate the stability of the initial-boundary approximation (2.1), one has to assure first the stability of scheme (1.8) should it be applied to the pure initial-value problem, $-\infty<x<\infty$.

ASSUMPTION III (Assumption 5.1 in [6]).

The scheme (1.8) is a stable approximation for the Cauchy problem, $-\infty<\nu<\infty$.

It is well-known (see for example [13, Section 4]) that Assumption III, may be characterized by the two following conditions:

(i) The von Neumann condition; namely, the z-solutions for the eigenvalue problem

$$(2.4a) \qquad \hat{Q}_{-1}(i\xi) - \sum_{\sigma=0}^{s} z^{-\sigma-1}\hat{Q}_\sigma(i\xi) = 0, \quad \hat{Q}(i\xi) \equiv \sum_{j=-r}^{p} a_{j\sigma}e^{ij\xi},$$

which may be rewritten in the equivalent form

$$(2.4b) \qquad \sum_{j=-r}^{p} a_j(z)e^{ij\xi} = 0; \quad a_j(z) = -\sum_{\sigma=0}^{s} z^{-\sigma-1}a_{j\sigma} + a_{j,-1},$$

satisfy $|z \equiv z(\xi)| \leq 1$ for $|\xi| \leq \pi$.

(ii) The solutions for the eigenvalue problem (2.4) which are lying on the unit circle, are simple, i.e., for $z_0 = z_0(\xi_0)$, $|z_0| = 1$, satisfying (2.4), we have

$$(2.5) \qquad \sum_{j=-r}^{p} a_j{}'(z)e^{ij\xi_0}\Big|_{z=z_0} \neq 0, \quad [\quad]' \equiv \frac{d[\quad]}{dz} .$$

In addition we require

<u>ASSUMPTION IV</u> (Assumption 5.4 in [6]).

Denote by $z_j(\xi)$ the solutions for (2.4). Then, the scheme (1.8) is either dissipative, i.e.,

$$|z_j(\xi)| < 1 \quad , \quad 0 < |\xi| \leq \pi,$$

or, it is nondissipative, i.e.,

$$|z_j(\xi)| = 1 \quad , \quad 0 \leq |\xi| \leq \pi.$$

Operating with our *scalar* approximation under Assumptions I-IV, enables us to use the results obtained by Gustafsson et. al.[6]; in particular, we are interested in their main result, characterizing the stability of an initial-boundary approximation. The remainder of this section is therefore devoted to a brief survey of some of the points concerning this matter.

Consider the z-eigenvalue problem -- the associated *resolvent equation* -- given by

$$(2.6) \qquad G(z)\Phi = 0, \quad G(z) \equiv G_{-1} - \sum_{\sigma=0}^{\max(\tau,s)} z^{-\sigma-1}G_\sigma,$$

which follows upon substituting a grid solution of the form $v_\nu(t) = z^{t/\Delta t}\Phi_\nu, \nu \geq -r+1$, into the approximation (2.1).

DEFINITION 2.2.

A complex number z_0, is said to be a spectrum-point of the approximation with associated eigenvector Φ, $\Phi \not\equiv 0$, if there exists a sequence of vectors $\Phi^{(j)}, j=1,2,\ldots,\Phi^{(j)} \in \ell_2(x)$, satisfying

$$(2.7) \qquad G(z_0)\Phi^{(j)} \xrightarrow{\|\ \|} 0, \quad \Phi^{(j)} \xrightarrow{\|\ \|} \Phi; \ j \longrightarrow \infty.$$

We note that the eigenvector Φ, associated with the spectrum point z_0, is not necessarily in $\ell_2(x)$. In case the vector Φ is indeed in $\ell_2(x)$, the point z_0 is an *eigenvalue* of the approximation; otherwise, when $\Phi \notin \ell_2(x)$, z_0 is a *generalized eigenvalue* of the approximation. In either case, the boundedness of the operator $G(z_0)$ implies that the corresponding eigenvector Φ satisfies

$$(2.8) \qquad G(z_0)\Phi = 0.$$

It is not hard to see that in order to assure the stability of the difference approximation, we have to assure first that the necessary condition of Ryabenkii-Goudunov is to be fulfilled, namely, that the approximation has no eigenvalues z with $|z| > 1$ (Lemma 4.1 in [6]). Indeed, the existence of such an eigenvalue z_0, $|z_0| > 1$, with associated eigenvector Φ, $\Phi \in \ell_2(x)$, implies that the grid function

$$(2.9) \qquad v_\nu(t) = z_0^{t/\Delta t}\,\Phi_\nu\,, \quad \nu \geq -r+1,$$

is a solution for the approximation (1.8), (1.12), with initial values $v_\nu(\sigma\Delta t) = z_0^\sigma \phi_\nu$, $\nu \geq -r+1$, $\sigma = 0,1,\ldots,s$, a solution which *exponentially diverges* with the refinement of the grid as $\Delta t \longrightarrow 0$. Evidently, such a divergence cannot be allowed within the limits of *any* stability definition and in particular definition 2.1, (Theorem 3.1 in [6]).

The main result in [6] strengthens the necessary condition mentioned above to be also sufficient.

THEOREM 2.1. (Theorem 5.1 in [6]).

The difference approximation (1.8), (1.12) is stable, if and only if it has neither eigenvalues nor generalized eigenvalues z, with $|z| \geq 1$.

2.2. A determinantal stability condition

In this section, we intend to express the stability condition which was given in Theorem 2.1, in a suitable *algebraic* formulation.

For that purpose, we consider the *characteristic equation*, associated with the basic scheme (1.8),

(2.10) $$P(z,\kappa) \equiv \sum_{j=-r}^{p} a_j(z)\kappa^j = 0,$$

whose $r+p$ roots κ_i counted according to their multiplicities, are continuous functions of z.

The behavior of these z-dependent solutions, plays a central role in determining the set of the spectrum points of approximation (1.8), (1.12). The following lemma summarizes the results which were given in [6] and [8] concerning those solutions, for *any solvable approximation*.

LEMMA 2.1.

Consider the solvable approximation (1.8), (1.12).

I. (Lemma 5.2 in [6]). When scheme (1.8) satisfies the von Neumann condition, then the $r+p$ solutions of its associated characteristic equation (2.10) are split for $|z| > 1$:

p with $|\kappa_i(z)| > 1$, and the rest r solutions with $0 < |\kappa_i(z)| < 1$.

II. (Lemma 2 in [8]). When scheme (1.8) is of dissipative type and it additionally satisfies

(2.11) $$\sum_{j=-r}^{p} a_j(z=e^{i\varphi}) \neq 0 \ , \quad 0 < |\varphi| \leq \pi \ ,$$

then the above splitting property is maintained for $|z| \geq 1, z \neq 1$.

PROOF The proof of both parts of the theorem is based on the idea of identifying the z-values for which the characteristic equation (2.10) has no κ-solutions on the unit circle, i.e., $\kappa = e^{i\xi}$, $0 \leq |\xi| \leq \pi$.

We first note, that the solutions $\kappa_i(z)$, $|z| \geq 1$ of (2.10), are exactly those which solve

(2.12) $$\kappa^r \cdot P(z,\kappa) \equiv \sum_{j=-r}^{p} a_j(z)\kappa^{j+r} = 0;$$

indeed, multiplying the characteristic equation (2.10) by the factor κ^r as in (2.12), does not yield additional zero solutions, $\kappa = 0$, since by Assumption I,

$$\kappa^r \cdot P(z,\kappa)_{|\kappa=0} = a_{-r}(z) \neq 0, \quad |z| \geq 1.$$

Now, concerning the first case, the von Neumann condition implies that the characteristic equation (2.12) has no κ-solutions on the unit circle for all z-values with $|z| > 1$, since by (2.4b) we have for $|z| > 1$

(2.13) $$e^{ir\xi} \cdot P(z,\kappa=e^{i\xi}) \equiv \sum_{j=-r}^{p} a_j(z)e^{i(j+r)\xi} \neq 0, \quad 0 \leq |\xi| \leq \pi.$$

Concerning the second dissipative case, a slight change is needed in Kreiss' original proof [8, Lemma 2] in order that the splitting property will be valid for our *mulit-level* scheme (1.8).

In this case, the dissipativity property, which indicates that the solutions $z(\xi) \equiv z(\kappa = e^{i\xi})$ of (2.12) satisfy

(2.14) $$|z(\xi)| < 1 \ , \ 0 < |\xi| \leq \pi,$$

implies that for $|z| \geq 1$, we have

$$(2.15) \qquad e^{ir\xi} \cdot P(z,\kappa=e^{i\xi}) \equiv \sum_{j=-r}^{p} a_j(z) e^{i(j+r)\xi} \neq 0 \ , \ 0 < |\xi| \leq \pi .$$

It therefore remains to check the single point $\xi = 0$.

By continuity considerations, it follows from (2.14) that

$$(2.16) \qquad |z(\xi=0)| \leq 1.$$

We recall our hypothesis (2.11), which implies that the solutions $z \equiv z(\xi)$ of (2.12) for $\kappa = e^{i\xi}|_{\xi=0}$ satisfy

$$(2.17) \qquad z(\xi=0) \neq e^{i\varphi} \ , \ 0 < |\varphi| \leq \pi .$$

Thus, by (2.16), (2.17), we obtain for $|z| \geq 1$, $z \neq 1$

$$(2.18) \qquad e^{ir\xi} \cdot P(z,\kappa=e^{i\xi})|_{\xi=0} \equiv \sum_{j=-r}^{p} a_j(z) \neq 0.$$

Combining (2.15) and (2.18) yields that (2.13) is valid in the second case, for all the z-values satisfying $|z| \geq 1$, $z \neq 1$.

Now, the solutions, $\kappa_i(z)$, for (2.12), are continuous functions of z. Therefore, the number of solutions κ_i satisfying $0 < |\kappa_i(z)| < 1$, is independent of z, as long as $|z| > 1$ in the first case, and $|z| \geq 1$, $z \neq 1$, in the second one. By letting z tend to infinity, $|z| \longrightarrow \infty$, it follows that this number is equal to the number of solutions κ inside the unit disc of

$$(2.19) \qquad \kappa^r \cdot P(|z| \longrightarrow \infty, \kappa) \equiv \kappa^r \cdot Q_{-1}(\kappa) \equiv \sum_{j=-r}^{p} a_{j,-1}\kappa^{j+r} = 0.$$

In order to find how the solutions of (2.19) are split we denote, as in (1.16), by r_0, $0 \leq r_0 \leq r$, the maximal index for which $a_{-r_0,-1} \neq 0$, and rewrite (2.19) in the form

$$\kappa^{r-r_0} \cdot \sum_{j=-r_0}^{p} a_{j,-1}\kappa^{j+r_0} = 0.$$

Then, the number of solutions κ, $0 \leq |\kappa| < 1$ of (2.19), consists of $r-r_0$ zero solutions, $\kappa = 0$, and, by Lemma 1.1 which is valid under our solvability assumption, r_0 additional solutions of

$$\sum_{j=-r_0}^{p} a_{j,-1}\kappa^{j} = 0.$$

Hence, there exist r solutions inside the unit disc, and p outside it, and the result follows.

<u>REMARK 2.1</u>. We note that in the course of proving Lemma 2.1, we didn't need Assumption I, except to assure that multiplying the characteristic equation (2.10) by the factor κ^r, as given in (2.12), does not yield additional zero solutions, $\kappa = 0$.

<u>REMARK 2.2</u>. To assure the splitting property for $|z| \geq 1$, $z \neq 1$ in the dissipative case, then according to the second part of the above lemma,

condition (2.11) has to hold. We note that this condition is actually part of the splitting property since violation of (2.11) means that for some φ_0, $0 < |\varphi_0| \le \pi$, we have

$$P(z=e^{i\varphi_0}, \kappa=1) = \sum_{j=-r}^{p} a_j(z=e^{i\varphi_0}) = 0;$$

that is, if (2.11) is violated then for some z with $|z| = 1$, $z \ne 1$, the characteristic equation (2.10) has a root ($\kappa=1$) on the unit circle. Moreover, condition (2.11) is generally necessary in the sense that it is independent of dissipativity. Indeed, let

$$v_\nu(t+\Delta t) = \sum_{j=-r}^{p} a_{j,0} v_\nu(t) \ , \ \nu = 1,2,\ldots,$$

be any two-level dissipative scheme which is (at least) zero order accurate, i.e., by (1.10a) we have

(2.21) $$\sum_{j=-r}^{p} a_j(z=1) = 1 - \sum_{j=-r}^{p} a_{j,0} = 0.$$

Now, let s be a positive integer and consider the solvable scheme

(2.22) $$v_\nu(t+\Delta t) = \sum_{j=-r}^{p} a_{j,0} v_\nu(t-s\Delta t) \ , \ \nu = 1,2,\ldots, \ .$$

This scheme is dissipative yet (2.11) is violated since by (2.21) we have for $z = \omega_j$, $\omega_j = e^{2\pi ij/(s+1)}$, $j = 1,2,\ldots,s$

$$(2.23) \qquad \sum_{j=-r}^{p} a_j(z)\big|_{z=\omega_j} \equiv 1-z^{-(s+1)} \cdot \sum_{j=-r}^{p} a_{j,0}\big|_{z=\omega_j} = 1 - \sum_{j=-r}^{p} a_{j,0} = 0.$$

The last example (2.22), which shows the necessity of the additional condition (2.11), is of course a degenerated one. When we turn to examine either the two-level or the three-level schemes, i.e., $s = 0$ or $s = 1$, which are apparently the ones used most often, we find that the additional condition (2.11), may be omitted, or, at least may be weakened. This is the content of the next lemma.

LEMMA 2.2.

Let the scheme (1.8) be accurate of (at least) order zero (that is, even the consistency, (1.10), is not necessarily required).

I. *For two-level scheme,* $s = 0$, *we have*

$$(2.24) \qquad \sum_{j=-r}^{p} a_j(z=e^{i\varphi}) \neq 0 \ , \ 0 < |\varphi| \leq \pi.$$

II. *For three-level scheme,* $s = 1$, *which satisfies*

$$(2.25) \qquad \sum_{j=-r}^{p} a_j(z=-1) \neq 0,$$

(2.24) *still holds.*

Thus, the additional condition (2.11) is automatically fulfilled in the two-level case, and has to be verified at the single point $z = -1$ in the three-level one.

PROOF. When scheme (1.8) is of (at least) zero order accuracy, (1.10a) implies

$$P(z=1,\kappa=1) = \sum_{j=-r}^{p} a_j(z=1) = 0. \tag{2.26}$$

Now, in the two-leveled case, the characteristic function

$$P(z,\kappa=1) \equiv Q_{-1}(\kappa=1) - z^{-1}\cdot Q_0(\kappa=1), \tag{2.27}$$

is a polynomial of *first* degree in the argument z^{-1}, and, by (2.26), its only root is $z^{-1} = 1$; hence

$$P(z=e^{i\varphi},\kappa=1) \equiv \sum_{j=-r}^{p} a_j(z=e^{i\varphi}) \neq 0 \ , \ 0 < |\varphi| \leq \pi,$$

thus, (2.24) holds.

In the three-leveled case, the characteristic function

$$P(z,\kappa=1) \equiv Q_{-1}(\kappa=1)-z^{-1}\cdot Q_0(\kappa=1) - z^{-2}\cdot Q_1(\kappa=1) \tag{2.28}$$

is a polynomial of *second* degree in the argument z^{-1}, whose coefficients are real, and by (2.26), $z^{-1} = 1$, is one of its two roots. Hence, the other root of (2.28) is real, and therefore

$$P(z=e^{i\varphi},\kappa=1) \equiv \sum_{j=-r}^{p} a_j(z=e^{i\varphi}) \neq 0 \ , \ 0 < |\varphi| < \pi \ . \tag{2.29}$$

Now, combining (2.29) with our hypothesis (2.25) which merely asserts that (2.29) is valid also for $|\varphi| = \pi$, gives the desired result.

We return now to the characteristic equation (2.10), the solutions of which were discussed in the Lemma 2.1.

Denote by $\kappa_\ell = \kappa_\ell(z)$, the *distinct, z-dependent* solutions of the characteristic equation (2.10), each with corresponding multiplicity $m_\ell = m_\ell(z)$. Since our basic scheme (1.8) is always assumed to satisfy the von Neumann condition (Assumption III), then by the splitting property stated in Lemma 2.1, we may distinguish between two groups of solutions of (2.10):

the group of the *inner solutions*, $\kappa_\ell(z)$, $1 \le \ell \le n$, which are characterized by

$$0 < |\kappa_\ell(z)| < 1 \quad , \quad |z| > 1;$$

and the group of the *outer solutions*, containing the rest of the solutions, and characterized by

$$|\kappa_\ell(z)| > 1 \ , \quad |z| > 1.$$

Note that by continuity, the inner (outer) solutions are well-defined for $|z| \ge 1$, where the milder inequalities $|\kappa_\ell| \le 1$ $(|\kappa_\ell| \ge 1)$ are valid.

Now, let z with $|z| \ge 1$ be given. If z is an eigenvalue or a generalized eigenvalue of the approximation, then there exists a corresponding (nontrivial) eigenvector Φ such that $G(z)\Phi = 0$; thus by the definition of $G(z)$ in (2.6), Φ must first satisfy the basic scheme associated with the resolvent equation

(2.30) $$(Q_{-1} - \sum_{\sigma=0}^{s} z^{-\sigma-1} Q_\sigma)\Phi_\nu = 0, \ \nu = 1,2,3,\dots .$$

Equation (2.30) is an ordinary difference equation with constant coefficients; hence, the most general form of an *eigenvector* Φ satisfying (2.30) is given by

$$\phi_\nu = \sum_{\ell=1}^{n} \sum_{k=0}^{m_\ell - 1} \sigma_{\ell,k} P_{\ell,k}(\nu) \kappa_\ell^\nu \ , \quad \nu \geq -r+1. \tag{2.31}$$

Here, $\kappa_\ell \equiv \kappa_\ell(z)$, are the distinct *inner* solutions of the characteristic equation (2.10) each with corresponding multiplicity $m_\ell \equiv m_\ell(z)$; $P_{\ell,k}(\nu)$ are *arbitrary* polynomials in ν with $\deg[P_{\ell,k}(\nu)] = k$; and $\sigma_{\ell,k}$ are free parameters to be determined, where by Lemma 2.1, their precise number is

$$\sum_{\ell=1}^{n} m_\ell = r.$$

<u>REMARK 2.3</u>. We note that the splitting property mentioned in Lemma 2.1, implies that for $|z| > 1$, the inner solutions, $\kappa_\ell(z)$, satisfy the strict inequality $|\kappa_\ell(z)| < 1$, hence, the eigenvector Φ given in (2.31) is in $\ell_2(x)$. Thus, the existence of a *generalized* eigenvalue z, is possible only for z lying on the unit circle, $|z| = 1$. Furthermore, operating under hypothesis (2.11), this possibility is reduced in the dissipative case, to the single point, $z = 1$, since the splitting property in this case is maintained also for $|z| = 1$, $z \neq 1$.

We now make a particular choice of the polynomials $P_{\ell,k}(\nu)$ in (2.31), which later proved useful. We choose

$$P_{\ell,k}(\nu) = \kappa_\ell^{-k} \, k! \, \binom{\nu}{k} \ ,$$

so that the most general solution of (2.30) which is used as eigenvector of the approximation, is of the form

(2.32) $$\phi_\nu = \sum_{\ell=1}^{n} \sum_{k=0}^{m_\ell - 1} \sigma_{\ell,k} k! \binom{\nu}{k} \kappa_\ell^{\nu-k}, \quad \nu \geq -r+1.$$

To determine the parameters $\sigma_{\ell,k}$, we recall that being an eigenvector of G, Φ must also satisfy the boundary conditions

(2.33) $$(G(z)\phi)_\mu = 0 \ , \ \mu = 0,-1,\ldots,-r+1.$$

The operator $G(z)$ is defined with the aid of the operators G_σ, whose operation at the boundary points is given by (1.12); hence (2.33) becomes

(2.34) $$(S_{-1}^{(\mu)} - \sum_{\sigma=0}^{\tau} z^{-\sigma-1} S_\sigma^{(\mu)})\phi_\mu = 0, \ \mu = 0,-1,\ldots,-r+1.$$

Inserting (2.32) in (2.34), we finally obtain

(2.35) $$\sum_{\ell=1}^{n} \sum_{k=0}^{m_\ell - 1} \sum_{j=0}^{q} [c_{j,-1}^{(\mu)} - \sum_{\sigma=0}^{\tau} z^{-\sigma-1} c_{j\sigma}^{(\mu)}] k! \binom{\mu+j}{k} \kappa_\ell^{\mu+j-k} \sigma_{\ell,k} = 0,$$

$$\mu = 0,-1,\ldots,-r+1,$$

which constitutes a linear homogeneous system of r equations in the r unknowns $\sigma_{\ell,k}$. Clearly, Φ is an eigenvector of the approximation, if and only if not all the $\sigma_{\ell,k}$ in (2.35) vanish, that is, (2.35) has a nontrivial solution.

At this point, we associate with the boundary conditions (1.12) a set of rational *boundary-functions*

(2.36) $$R_\mu(z,\kappa) \equiv \sum_{j=0}^{q} [c_{j,-1}^{(\mu)} - \sum_{\sigma=0}^{\tau} z^{-\sigma-1} c_{j\sigma}^{(\mu)}]\kappa^{\mu+j}, \quad \mu = 0,-1,\ldots,-r+1,$$

which are *uniquely determined by the boundary coefficients* $c_{j\sigma}^{(\mu)}$ Upon defining the scalar functions

(2.37a) $$S_\sigma^{(\mu)}(\kappa) \equiv \sum_{j=0}^{q} c_{j\sigma}^{(\mu)}\kappa^{\mu+j}, \quad \sigma = -1,0,\ldots,\tau$$

the associated boundary-functions may be rewritten as

(2.37b) $$R_\mu(z,\kappa) = S_{-1}^{(\mu)}(\kappa) - \sum_{\sigma=0}^{\tau} z^{-\sigma-1} S_\sigma^{(\mu)}(\kappa), \quad \mu = 0,-1,\ldots,-r+1.$$

Since

$$\frac{\partial^k R_\mu(z,k)}{\partial\kappa^k} = \sum_{j=0}^{q} [c_{j,-1}^{(\mu)} - \sum_{\sigma=0}^{\tau} z^{-\sigma-1} c_{j\sigma}^{(\mu)}]k!\binom{\mu+j}{k}\kappa^{\mu+j-k},$$

the system (2.35) is cast in the form

(2.38) $$\sum_{\ell=1}^{n} \sum_{k=0}^{m_\ell-1} \frac{\partial^k R_\mu(z,\kappa)}{\partial\kappa^k} \sigma_{\ell,k} = 0, \quad \mu = 0,-1,\ldots,-r+1.$$

It follows that the coefficient matrix of this system, which we denote by

(2.39a) $$D = D(z;\kappa_1,\ldots,\kappa_n;m_1,\ldots,m_n)$$

is of the form

$$D = [B(z,\kappa_1,m_1),\ B(z,\kappa_2,m_2),\ldots,B(z,\kappa_n,m_n)],$$

where $B(z,\kappa_\ell,m_\ell)$, $1 \le \ell \le n$ are $r\times m_\ell$ dimensional blocks given by

(2.39c)
$$B(z,\kappa_\ell,m_\ell) = \left[\begin{bmatrix} R_0(z,\kappa) \\ R_{-1}(z,\kappa) \\ \vdots \\ R_{-r+1}(z,\kappa)\end{bmatrix}, \frac{\partial}{\partial\kappa}\begin{bmatrix} R_0(z,\kappa) \\ R_{-1}(z,\kappa) \\ \vdots \\ R_{-r+1}(z,\kappa)\end{bmatrix},\ldots,\frac{\partial^{m_\ell-1}}{\partial\kappa^{m_\ell-1}}\begin{bmatrix} R_0(z,\kappa) \\ R_{-1}(z,\kappa) \\ \vdots \\ R_{-r+1}(z,\kappa)\end{bmatrix}\right]_{|\kappa=\kappa_\ell}, 1\le\ell\le n.$$

We recall that Φ is an eigenvector of the approximation if and only if (2.38) has a nontrivial solution, i.e., if D is singular. This gives us

LEMMA 2.3.

Let z with $|z| \ge 1$ be given, and let $\kappa_\ell = \kappa_\ell(z)$, $1 \le \ell \le n$, be the corresponding distinct inner solutions of the characteristic equation (2.10), each with multiplicity $m_\ell = m_\ell(z)$.
Then z is an eigenvalue or a generalized eigenvalue of the approximation if and only if

$$\det[D(z;\kappa_1,\ldots,\kappa_n;m_1,\ldots,m_n)] = 0.$$

By Theorem 2.1, the stability of the approximation is assured if and only if it has no eigenvalues nor generalized eigenvalues z with $|z| \ge 1$; so by applying

Lemma 2.3 we finally obtain an algebraic formulation of the stability condition of the type which we look for.

THEOREM 2.2.

The difference approximation (1.8), (1.12) of the initial-boundary value problem (1.7) is stable, if and only if for every z $|z| \geq 1$, with distinct inner solutions κ_ℓ, $1 \leq \ell \leq n$, each with multiplicity m_ℓ

$$\det[D(z;\kappa_1,\ldots,\kappa_n;m_1,\ldots,m_n)] \neq 0.$$

Theorem 2.2 is simplified when the boundary conditions are of one-level type, (see (1.12b)), i.e.,

$$S_{-1}^{(\mu)} v_\mu(t+\Delta t) = 0, \quad \mu = 0,-1,\ldots,-r+1. \tag{2.40}$$

In this case the associated boundary-functions (2.37) are

$$R_\mu(\kappa) \equiv S_{-1}^{(\mu)}(\kappa) \equiv \sum_{j=0}^{q} c_{j,-1}^{(\mu)} \kappa^{\mu+j}$$

and the matrix coefficient D in (2.39) is given by

$$D = D(\kappa_1,\ldots,\kappa_n;m_1,\ldots,m_n) \equiv [B(\kappa_1,m_1),\ldots,B(\kappa_n,m_n)] \tag{2.41a}$$

with

(2.41b)
$$B(\kappa_\ell,m_\ell) = \left[\begin{bmatrix} R_0(\kappa) \\ R_{-1}(\kappa) \\ \vdots \\ R_{-r+1}(\kappa) \end{bmatrix}, \frac{d}{d\kappa}\begin{bmatrix} R_0(\kappa) \\ R_{-1}(\kappa) \\ \\ R_{-r+1}(\kappa) \end{bmatrix}, \ldots, \frac{d^{m_\ell-1}}{d\kappa^{m_\ell-1}}\begin{bmatrix} R_0(\kappa) \\ R_{-1}(\kappa) \\ \\ R_{-r+1}(\kappa) \end{bmatrix}\right]_{\kappa=\kappa_\ell}, \quad 1 \leq \ell \leq n.$$

The matrix D in (2.41) no longer depends *explicitly* on z, but via the inner solutions $\kappa_\ell = \kappa_\ell(z)$ and their multiplicities $m_\ell = m_\ell(z)$, hence Theorem 2.2 becomes

COROLLARY 2.1.

The difference approximation (1.8), (1.12b) *is stable, if and only if for every* z, $|z| \geq 1$ *with distinct inner solutions* κ_ℓ, $1 \leq \ell \leq n$, *each with multiplicity* m_ℓ

$$\det[D(\kappa_1,\ldots,\kappa_n;\ m_1,\ldots,m_n)] \neq 0.$$

3. SCHEME-INDEPENDENT STABILITY CRITERIA

3.1. Translatory boundary conditions — the determinant condition

In the previous chapters, we dealt with boundary conditions of the general form (1.12), where each boundary value, $v_\mu(t+\Delta t)$, is determined by a linear combination of computed grid values which is dependent on the position of the boundary value to be computed.

In this chapter we start discussing *translatory* boundary conditions; that is, the *same* linear combination is used to compute the boundary values $v_\mu(t+\Delta t)$, $\mu = 0,-1,\ldots,-r+1$, independently of their position. In other words, the translatory case is characterized by applying a repeated procedure, where the computation is done by translating the *same* linear combination in the usual specified order, namely, $\mu = 0,-1,\ldots,-r+1$. The translatory boundary conditions are thus cast in the form

$$S_{-1}v_\mu(t+\Delta t) = \sum_{\sigma=0}^{\tau} S_\sigma v_\mu(t-\sigma\Delta t),\ \mu = 0,-1,\ldots,-r+1,\ t \geq \tau\Delta t$$

$$S_\sigma = \sum_{j=0}^{q} c_{j\sigma}E^j\ ,\ \sigma = -1,0,\ldots,\tau \tag{3.1}$$

where the coefficients $c_{j\sigma}$ are no longer dependent on μ.

We note that when the discrete boundary domain is reduced to the single point x_0, such as in the case of the widely used 3-point schemes (i.e., $r = p = 1$), the computation at the boundary is of translatory type by definition.

Hereafter, we concentrate on searching for conditions assuring stability in the translatory case.

The rational boundary-functions associated with (3.1) are

(3.2a) $$R_\mu(z,\kappa) = \sum_{j=0}^{q} c_j(z)\kappa^{\mu+j}, \mu = 0,-1,\ldots,-r+1,$$

where the scalar functions $c_j(z)$ are given by

(3.2b) $$c_j(z) = -\sum_{\sigma=0}^{\tau} z^{-\sigma-1} c_{j\sigma} + c_{j,-1}, \; 0 \le j \le q.$$

In particular, for $\mu = 0$ we get

(3.3a) $$R_0(z,\kappa) = \sum_{j=0}^{q} c_j(z)\kappa^j \equiv S_{-1}(\kappa) - \sum_{\sigma=0}^{\tau} z^{-\sigma-1} S_\sigma(\kappa),$$

where the scalar functions $S_\sigma(\kappa)$, are given by

(3.3b) $$S_\sigma(\kappa) = \sum_{j=0}^{q} c_{j\sigma}\kappa^j, \; \sigma = -1,0,\ldots,\tau.$$

By (3.2a) and (3.3a), we have

$$R_\mu(z,\kappa) \equiv \kappa^\mu R_0(z,\kappa), \; \mu = 0,-1,\ldots,-r+1,$$

so the $r \times r$ matrix

$$D \equiv D(z,\kappa_1,\ldots,\kappa_n;m_1,\ldots,m_n) \equiv [B(z,\kappa_1,m_1),\ldots,B(z,\kappa_n,m_n)]$$

of (2.39) is given by the $r \times m_\ell$ dimensional blocks

$$B(z,\kappa_\ell,m_\ell) = \left[\begin{bmatrix} R_0(z,\kappa) \\ \kappa^{-1}R_0(z,\kappa) \\ \vdots \\ \kappa^{-r+1}R_0(z,\kappa) \end{bmatrix}, \frac{\partial}{\partial\kappa}\begin{bmatrix} R_0(z,\kappa) \\ \kappa^{-1}R_0(z,\kappa) \\ \vdots \\ \kappa^{-r+1}R_0(z,\kappa) \end{bmatrix}, \ldots, \frac{\partial^{m_\ell-1}}{\partial\kappa^{m_\ell-1}}\begin{bmatrix} R_0(z,\kappa) \\ \kappa^{-1}R_0(z,\kappa) \\ \vdots \\ \kappa^{-r+1}R_0(z,\kappa) \end{bmatrix}\right]_{\kappa=\kappa_\ell}, 1 \le \ell \le n.$$

The fact that D is determined now by the single boundary-function $R_0(z,\kappa)$ enables us to significantly simplify the stability condition given in Theorem 2.2 by replacing its determinantal criterion with the following *scalar* condition.

<u>THEOREM 3.1</u>.

<u>The difference approximation</u> (1.8), (3.1) <u>of the initial-boundary value problem</u> (1.7) <u>is stable, if and only if for every</u> z, $|z| \ge 1$, <u>with corresponding distinct inner solutions</u> κ_ℓ, $1 \le \ell \le n$, <u>we have</u>

$$(3.4) \qquad R_0(z,\kappa_\ell) \ne 0, \quad \ell = 1,2,\ldots,n.$$

<u>PROOF</u>. Suppose there exists z_0, $|z_0| \ge 1$, with corresponding inner solution $\kappa_{\ell_0} = \kappa_{\ell_0}(z)$, $|\kappa_{\ell_0}| \le 1$ violating (3.4) by satisfying

$$R_0(z_0,\kappa_{\ell_0}) = 0.$$

Then the left column of the block $B(z_0,\kappa_{\ell_0},m_{\ell_0})$ is identically zero, hence the matrix D is singular and by Theorem 2.2, the approximation is unstable.

Conversely, suppose (3.4) is valid, and we want to prove stability, where by Theorem 2.2 it suffices to show that for every z, $|z| \geq 1$, with distinct inner solutions κ_ℓ, $1 \leq \ell \leq n$, each with multiplicity m_ℓ, $1 \leq \ell \leq n$, we have

$$\det[D(z;\kappa_1,\ldots,\kappa_n;m_1,\ldots,m_n)] \neq 0.$$

For that purpose, let

$$(3.5) \qquad \sum_{\mu=-r+1}^{0} \alpha_\mu \begin{bmatrix} \kappa_1^\mu R_0(z,\kappa_1) \\ \frac{\partial}{\partial\kappa_1}[\kappa_1^\mu R_0(z,\kappa_1)] \\ \vdots \\ \frac{\partial^{m_n-1}}{\partial\kappa_n^{m_n-1}}[\kappa_n^\mu R_0(z,\kappa_n)] \end{bmatrix}' = 0$$

be a vanishing linear combination of the rows of D. The vector relation in (3.5) consists of the r scalar equations

$$\sum_{\mu=-r+1}^{0} \alpha_\mu \frac{\partial^k}{\partial\kappa_\ell^k}[\kappa_\ell^\mu R_0(z,\kappa_\ell)] = 0 \ ; \ 0 \leq k \leq m_\ell-1, \ 1 \leq \ell \leq n,$$

which we write as

$$(3.6) \qquad \frac{\partial^k}{\partial\kappa_\ell^k}[\kappa_\ell^{-r+1} R_0(z,\kappa_\ell)\cdot \sum_{\mu=-r+1}^{0} \alpha_\mu \kappa_\ell^{r+\mu-1}] = 0; \ 0 \leq k \leq m_\ell-1, \ 1 \leq \ell \leq n.$$

By our hypothesis (3.4), the left member in the above brackets satisfies

$$\kappa_\ell^{-r+1} R_0(z,\kappa_\ell) \neq 0 \ , \ 1 \leq \ell \leq n.$$

Thus, expanding by Leibniz' rule and using induction on $k \geq 0$, we find that the right member in (3.6) has vanishing derivatives at $\kappa = \kappa_\ell$; i.e.,

$$\frac{d^k}{d\kappa^k}\Big[\sum_{\mu=-r+1}^{0} \alpha_\mu \kappa^{r+\mu-1}\Big]\Big|_{\kappa=\kappa_\ell} = 0 \ ; \ 0 \leq k \leq m_\ell - 1, \ 1 \leq \ell \leq n.$$

We conclude that the polynomial

$$T(\kappa) = \sum_{\mu=-r+1}^{0} \alpha_\mu \kappa^{r+\mu-1}$$

which is of degree $r-1$ at most, has r roots; κ_ℓ, $1 \leq \ell \leq n$ each with multiplicity m_ℓ. Hence, $T(\kappa) \equiv 0$ and the coefficients α_μ must vanish. By (3.5), therefore, the rows of D are linearly independent, so the matrix is nonsingular and stability follows by Theorem 2.2.

As was realized in the previous section, if (3.1) is reduced to the one-level case

$$(3.7) \qquad S_{-1} v_\mu(t+\Delta t) = \sum_{j=0}^{q} c_{j,-1} v_{\mu+j}(t+\Delta t) = 0, \ \mu = 0,-1,\ldots,-r+1,$$

then the associated boundary-functions

$$R_\mu(k) = \sum_{j=0}^{q} c_{j,-1} \kappa^{\mu+j} \equiv \kappa^\mu S_{-1}(\kappa), \ \ \mu = 0,-1,\ldots,-r+1,$$

cease to depend *explicitly* on z, and Theorem 3.1 provide us with

COROLLARY 3.1.

The difference approximation (1.8), (3.7) is stable, if and only if for every z, $|z| \geq 1$, with corresponding distinct inner solutions κ_ℓ, $1 \leq \ell \leq n$, we have

$$R_0(\kappa_\ell) \equiv S_{-1}(\kappa_\ell) \neq 0, \quad \ell = 1,2,\ldots,n.$$

Finally, we note that in the case $r = 1$, which was mentioned in the beginning of this section as translatory one by definition, the matrix representation of the results which obtained in Theorem 2.2 and Corollary 2.2, reduced respectively to the scalar results given in Theorem 3.1 and Corollary 3.1. Indeed, in this case, the matrix D is the scalar boundary-function R_0.

3.2. Scheme-independent stability criteria

The stability criterion given in Theorem 3.1, involves the both parts which constitute the approximation (1.8), (3.1); these are the translatory boundary conditions (3.1) which generate the boundary-function $R_0(z,\kappa)$, and the (basic) scheme (1.8) which induces the characteristic equation (2.10) whose z-dependent solutions $\kappa_\ell = \kappa_\ell(z)$ are used as test points for the stability of the approximation.

Our main aim in this section is to provide stability criteria which do not take into account the (basic) scheme (1.8), but instead, *are given solely in terms of the boundary conditions*. In such a way, we shall be able to answer the question whether a given boundary treatment *violates* the stability of *any* basic scheme (as an approximation to the pure-initial problem-Assumption III).

<u>REMARK 3.1</u>. We emphasize that in the scheme-independent stability analysis carried out below, *it is always assumed that the (basic) schemes considered obey the four basic assumptions, Assumptions I-IV,* which were originally made in [6].

The larger part of this section discusses the (somewhat simpler) outflow case. We recall that these are the outflow scalar components, the stability of which we have to look for in order to assure the stability of the entire vector approximation. The end of this section is devoted to the inflow case, where the results follow easily, merely by updating the results previously obtained for the outflow problem.

We start by recalling Lemma 7 in [8] which discusses the behavior of the inner solutions $\kappa_\ell(z)$ in the neighbourhood of the point $z = 1$.

<u>LEMMA 3.1</u>. (Lemma 7 in [8])

<u>Consider the consistent (basic) scheme (1.8) as an approximation to the outflow,</u>

$a > 0$, (<u>inflow</u>, $a > 0$) <u>problem</u> (1.7). <u>Then, its associated characteristic equation</u> (2.10) <u>has exactly one outer</u> (<u>inner</u>) <u>solution</u> $\kappa \equiv \kappa(z)$ <u>which satisfies</u> $\kappa(z=1) = 1$.

<u>PROOF</u>. The consistency condition (1.10a) implies that $z = 1$ is a solution of the eigenvalue problem (2.4b)

$$\sum_{j=-r}^{p} a_j(z)e^{ij\xi}\Big|_{z=1,\xi=0} = 0,$$

and by part (ii) of Assumption III (see (2.5)), $z = 1$ must be a simple solution, i.e.,

$$\sum_{j=-r}^{p} a_j'(z=1) \neq 0 \ , \quad [\ \ ]' \equiv \frac{d[\ \ ]}{dz} \ . \tag{3.8}$$

By the consistency conditions (1.10a), (1.10b) and by (3.8) we have

$$P(z,\kappa)\Big|_{z=\kappa=1} = \sum_{j=-r}^{p} a_j(z)\Big|_{z=1} = 0, \tag{3.9a}$$

$$\frac{\partial P}{\partial \kappa}(z,\kappa)\Big|_{z=\kappa=1} = \sum_{j=-r}^{p} ja_j(z)\Big|_{z=1} = -\lambda a\cdot\sum_{j=-r}^{p} a_j'(z)\Big|_{z=1} \neq 0. \tag{3.9b}$$

Hence, we may apply the Implicit Function Theorem obtaining that in the neighbourhood of $z = 1$, the characteristic equation (2.10) can be uniquely solved for κ as a differential function of z; that is, there exists a single root-function of (2.10), $\kappa \equiv \kappa(z)$, which satisfies

$$\kappa(z=1) = 1. \tag{3.10a}$$

Applying the consistency condition (1.10b) once more (see (3.9b) yields

$$\frac{\partial P}{\partial \kappa}(z,\kappa)\Big|_{z=\kappa=1} = -\lambda a \cdot \sum_{j=-r}^{p} a_j'(z)\Big|_{z=1} = -\lambda a \cdot \frac{\partial P}{\partial z}(z,\kappa)\Big|_{z=\kappa=1} ,$$

hence the root-function $\kappa(z)$ determined above satisfies

$$\text{(3.10b)} \qquad \kappa'(z)\big|_{z=1} = -\frac{\partial P}{\partial z} \Big/ \frac{\partial P}{\partial \kappa}\Big|_{z=\kappa=1} = 1/\lambda a \ , \qquad [\ \ ]' \equiv \frac{d[\ \ ]}{dz} .$$

Combining (3.10a), (3.10b) implies that for $z = 1+\delta$, $\delta > 0$ sufficiently small, we have

$$\kappa(z) = 1+(\lambda a)^{-1}\delta + \mathcal{O}(\delta^2).$$

Hence, the inequality

$$\text{(3.11)} \qquad |\kappa(z)| > 1 \quad (|\kappa(z)| < 1) \ , \ a > 0 \ (a < 0),$$

holds in the right real neighbourhood of $z = 1$, and since the basic scheme (1.8) is assumed to satisfy the von Neumann conditions then by Lemma 2.1I this inequality holds for all z with $|z| > 1$. Thus $\kappa(z)$ is an outer (inner) solution according to the positive (negative) sign of the coefficient a which completes the proof.

In the course of our discussion about scheme-independent stability criteria, we introduce two additional assumptions complementing the first four already made. We will show that the new assumptions are necessary for stability and provide scheme-independent algebraic tests to verify their validity.

To introduce the first new assumption, let the scalar functions $a_j(z)$ and $c_j(z)$ be as in (1.11) and (3.2b), respectively.

ASSUMPTION V.

The z-function $\Delta(z)$, given by

$$\Delta(z) \equiv \left| \sum_{j=-r}^{p} a_j(z) \right| + \left| \sum_{j=0}^{q} c_j(z) \right|, \tag{3.12a}$$

which may be rewritten in the form

$$\Delta(z) \equiv |P(z,\kappa=1)| + |R_0(z,\kappa=1)|, \tag{3.12b}$$

satisfies

$$\Delta(z=e^{i\varphi}) \neq 0 \ , \ 0 < |\varphi| \leq \pi. \tag{3.12c}$$

REMARK 3.2. We note that ***Assumption V is not scheme-independent,*** since the function $\Delta(z)$ in (3.12) depends on both-on the coefficients $a_j(z)$ determined by the basic scheme (1.8) and on the coefficients $c_j(z)$ determined by the boundary conditions (3.1). However, it should be pointed out that although Assumption V depends on both parts of the approximation, its validity can be assured by considering only one of these two parts. That is, (3.12c) is valid if either the scheme-dependent condition

$$(3.13) \qquad P(z=e^{i\varphi},\kappa=1) \equiv \sum_{j=-r}^{p} a_j(z=e^{i\varphi}) \neq 0 \ , \ 0 < |\varphi| \leq \pi,$$

or the boundary-dependent condition

$$(3.14) \qquad R_0(z=e^{i\varphi},\kappa=1) \equiv \sum_{j=0}^{q} c_j(z=e^{i\varphi}) \neq 0 \ , \ 0 < |\varphi| \leq \pi,$$

holds. In particular, one may use (3.14) as a scheme-independent test to verify the validity of Assumption V.

Verifying the validity of Assumption V becomes much simpler when the basic scheme or the boundary conditions are either two-leveled or three-leveled, i.e., when s or τ obtain the values 0 or 1. This is the content of the next lemma.

LEMMA 3.2.

Each one of the following four conditions is sufficient for Assumption V to hold:

(i) - *The basic scheme* (1.8) *is two-leveled, i.e.,* $s = 0$.

(ii) - *The boundary conditions* (3.1) *are two-leveled, i.e.,* $\tau = 0$, *and are accurate of order (at least) zero.*

(iii) - *The basic scheme* (1.8) *is three-leveled, i.e.,* $s = 1$, *and in addition we have*

$$(3.15a) \qquad P(z=-1,\kappa=1) \equiv \sum_{j=-r}^{p} a_j(z=-1) \neq 0.$$

(iv) - *The boundary conditions* (3.1) *are three-leveled, i.e.,* $\tau = 1$, *are accurate of order (at least) zero and in addition we have*

$$(3.15b) \qquad R_0(z=-1,\kappa=1) \equiv \sum_{j=0}^{q} c_j(z=-1) \neq 0.$$

Thus, roughly speaking, Assumption V is automatically fulfilled in the two-level case, and has to be verified at the single point $z = -1$ in the three-level one.

PROOF. As explained in Remark 3.2 above, each one of the two conditions - either (3.13) or (3.14) is sufficient for Assumption V to hold; thus the result of our lemma follows directly from Lemma 2.2.

Indeed, in cases (i) and (iii) Lemma 2.2 implies that condition (2.24) or equivalently (3.13) holds and hence Assumption V is valid. In the remaining cases, (ii) and (iv), the accuracy hypothesis of the boundary conditions enables us to follow the proof of Lemma 2.2 replacing the functions $a_j(z)$ by $c_j(z)$ and obtaining that condition (3.14) holds. Hence, Assumption V is valid also in these cases.

We turn now to discuss scheme-independent stability criteria and let us start by studying schemes of dissipative type. The important point in the stability analysis of such schemes is the fact that *generalized* eigenvalues z with $|z| \geq 1$ may exist only at the single point $z = 1$ (see Remark 2.3). Recalling also Lemma 3.1, we find that for $|z| \geq 1$ all the corresponding inner solutions $\kappa_\ell(z)$ are lying *inside* the unit disc, i.e., $|\kappa_\ell(z)| < 1$. Indeed, this argument is the basis for our next theorem discussing scheme-independent stability criteria for *any* dissipative (basic) scheme.

THEOREM 3.2.

Consider the basic scheme (1.8) of dissipative type together with translatory boundary conditions as an approximation to the outflow problem (1.7).

For one-level boundary conditions, $\tau = -1$, we have

(I) - the difference approximation (1.8), (3.7) is stable if for every κ with

$0 < |\kappa| < 1,$

(3.16) $$R_0(\kappa) = \sum_{j=0}^{q} c_{j,-1}\kappa^j \neq 0.$$

<u>For multi-level boundary conditions</u>, $\tau > -1$, <u>we have</u>

(II) - <u>the difference approximation</u> (1.8), (3.1) <u>is stable for every</u> z <u>with</u> $|z| \geq 1$ <u>and every</u> κ <u>with</u> $0 < |\kappa| < 1$,

(3.17) $$R_0(z,\kappa) \equiv \sum_{j=0}^{q} c_j(z)\kappa^j \neq 0.$$

<u>PROOF</u>. Take an arbitrary z with $|z| \geq 1$, and let $\kappa_\ell(z)$, $0 < |\kappa_\ell(z)| \leq 1$, be *any* corresponding inner solution, so that

(3.18) $$P(z,\kappa_\ell) \equiv \sum_{j=-r}^{p} a_j(z)\kappa_\ell^j = 0.$$

In order to assure stability, it suffices, according to Theorem 3.1, to show that

(3.19a) $$R_0(z,\kappa_\ell) \neq 0.$$

In particular, concerning the one-level case $\tau = -1$, the boundary-function R_0 does no longer explicitly depend on z and the sufficient condition (3.19a) is cast in the form (see Corollary 3.1).

(3.19b) $$R_0(\kappa_\ell) \neq 0.$$

For the case where the inner solution $\kappa_\ell(z)$ is inside the unit disc, i.e., $0 < |\kappa_\ell(z)| < 1$, (3.19a) and (3.19b) follow respectively by hypothesis (3.16) and (3.17).

Let us consider then the case where the inner solution $\kappa_\ell(z)$ is lying on the unit circle, i.e., $\kappa_\ell(z) = e^{i\xi}$, $0 \leq |\xi| \leq \pi$.

Our assumption of the dissipativity of the basic scheme means that the z-values which satisfy (3.18) with inner solution of the form $\kappa_\ell = e^{i\xi}$, $0 < |\xi| \leq \pi$, obey the inequality

$$|z(\kappa_\ell = e^{i\xi})| < 1, \quad 0 < |\xi| \leq \pi. \tag{3.20}$$

Hence, the only possibility to satisfy (3.18) by an inner solution of the form $\kappa_\ell = e^{i\xi}$ and by z with $|z| \geq 1$, is the possibility of $\kappa_\ell = e^{i\xi}|_{\xi=0} = 1$, where by (3.20), continuity implies that the corresponding z-value satisfies

$$|z(\kappa_\ell = e^{i\xi}|_{\xi=0})| \leq 1. \tag{3.21}$$

We therefore conclude that it remains to verify (3.19) in the case where $\kappa_\ell = 1$ and the corresponding z-values are lying on the unit circle, $z = e^{i\varphi}$, $0 \leq |\varphi| \leq \pi$.

Now, since the dissipative scheme (1.8) is consistent with the *outflow* problem (1.7), then according to Lemma 3.1, $\kappa_\ell = 1$ is excluded as an *inner* solution corresponding to $z = 1$. For the remaining z-values, $z = e^{i\varphi}, 0 < |\varphi| \leq \pi$, which may be taken into consideration, we have by (3.18) and by Assumption V

$$(3.22) \qquad |R_0(z=e^{i\varphi},\kappa_\ell=1)| = \Delta(z=e^{i\varphi}) \neq 0 \ , \ 0 < |\varphi| \leq \pi \ .$$

That is, (3.19) is valid also in this case, and stability follows.

The first part of Theorem 3.2 discussing one-level boundary conditions, provides a relatively simple stability criterion, as it depends on one variable, κ. Thus for example, m-order extrapolated boundary values

$$(3.23) \qquad (I-E)^{m+1} v_\mu = 0 \ , \ \mu = 0,-1,\ldots,-r+1$$

can be easily checked as satisfying (3.16), since we have

$$R_0(\kappa) \equiv (1-\kappa)^{m+1} \neq 0 \ , \ 0 < |\kappa| < 1.$$

In particular, in the case of a two-level basic scheme, where the validity of Assumption V follows by Lemma 3.2, we obtain the following well-known result [3], [7].

<u>COROLLARY 3.2</u>. (Theorem 5.2 in [7]).

<u>The two-level dissipative scheme</u> (1.8) <u>together with extrapolated boundary values as given in</u> (3.23) <u>constitute a stable approximation to the outflow problem</u> (1.7).

In the last corollary we required that the basic scheme will be two-leveled in order tc assure the validity of Assumption V. When we turn to extend Corollary 3.2 to the *general multi-level* case, we find that Assumption V is indeed a necessary one, as shown by the following example:

Consider the three-level 5-point dissipative basic scheme

$$(3.24) \quad v_\nu(t+\Delta t) = [I-\frac{\epsilon}{16}(E-I)^2(I-E^{-1})^2]v_\nu(t-\Delta t) + \lambda a(E-E^{-1})v_\nu(t),$$

$$\nu = 1,2,\dots, \quad \lambda a \le 1-\epsilon, \ \epsilon < 1,$$

which follows by adding the dissipative term $-\frac{\epsilon}{16}(E-I)^2(I-E^{-1})^2 v_\nu(t-\Delta t)$, to the usual Leap-Frog scheme, see [11, Section 9]. For both schemes, the associated characteristic equation has exactly one inner solution $\kappa = \kappa(z)$, satisfying $\kappa(z=-1) = 1$ (see [6, Lemma 6.2]). Now, when the scheme is complemented by m-order extrapolation of the boundary values $v_0(t), v_{-1}(t)$, the approximation becomes unstable, as follows from Corollary 3.1, since we have

$$R_0[\kappa(z)]_{|z=-1} = (1-\kappa)^{m+1}_{|\kappa=1} = 0.$$

This instability is explained by the fact that the approximation fails to satisfy Assumption V. Indeed, in the general case of one-level boundary conditions which are at least zero-order accurate, such as the m-order extrapolation, we have by (1.10a)

$$\sum_{j=0}^{q} c_j(z) \equiv \sum_{j=0}^{q} c_{j,-1} = 0.$$

Hence, Assumption V, which becomes

$$\Delta(z) \equiv \left| \sum_{j=-r}^{p} a_j(z) \right| \neq 0 \; ; \; z = e^{i\varphi}, \; 0 < |\varphi| \leq \pi ,$$

is violated by our scheme (3.24) at the point $z = -1$ since

$$\Delta(z)\big|_{z=-1} = \left| 1 - \frac{1}{z^2} \right|_{z=-1} = 0 .$$

We note that by Lemma 3.2, $z = -1$ is indeed the only possible point for a *three-level* scheme to violate Assumption V.

It was already mentioned, that the stability criterion (3.16) provided in Theorem 3.2I which discusses one-level boundary conditions, is relatively a simple one, since it involves only one variable , κ. Concerning the wide family of *multi-level* boundary conditions, the stability criterion provided by Theorem 3.2II in (3.17), is more complicated since it involves *two independent* variables, z and κ.

This motivates us to look for z-independent alternatives to Theorem 3.2II, for both, dissipative and particularly non-dissipative schemes, complemented by *multi-level* boundary conditions. This matter will occupy the remainder of our disscusion about the outflow problem.

We start by introducing the boundary-scheme associated with the boundary conditions (3.1)

$$(3.25) \qquad \begin{aligned} S_{-1} v_\nu(t+\Delta t) &= \sum_{\sigma=0}^{\tau} S_\sigma v_\nu(t-\sigma\Delta t), \quad \nu = 0,\pm 1,\pm 2,\ldots, \\ S_\sigma &= \sum_{j=0}^{q} c_{j\sigma} E^j \quad , \quad \sigma = -1,0,\ldots,\tau, \end{aligned}$$

which is generated by extending the definition of the boundary values in (3.1), v_μ, $\mu = 0,-1,\dots,-r+1$, to all grid points x_ν, $-\infty < \nu < \infty$.

Since the boundary conditions (3.1) were assumed to be multi-leveled, i.e., τ is non-negative, it follows that the boundary-scheme (3.25) is well-defined as a difference scheme, whose values are computed by *advancing in the direction of the time-axis.*

As was realized in Section 2.2, the splitting property described in lemma 2.1 was the key in investigating the solvable (-Assumption II), scheme (1.8), and we would like this result to be applicable also for the boundary-scheme (3.25). For that reason, we require the solvability of the boundary-scheme by making the following analogy of Assumption II.

ASSUMPTION VI.

The boundary-scheme (3.25) is solvable; that is, there exists a constant $K_0 > 0$, such that for every $\Psi \in \ell_2(x)$, there is a unique solution, w, $w \in \ell_2(x)$ for

(3.26a) $$S_{-1}w_\nu = \Psi_\nu \ , \ \nu = -r+1,-r+2,\dots,$$

with

(3.26b) $$\|w\|_x^2 \le K_0^2 \cdot \|\Psi\|_x^2 \ .$$

REMARK 3.3. To assure Assumption VI, one may use Lemma 1.1 which characterizes solvability. Recalling the notation there, the index r_0 in (1.16) equals zero in the case of the right-sided boundary-scheme (3.26a), since by (1.13) we have $c_{0,-1} \neq 0$. Thus, we may apply the second part of Lemma 1.1, replacing the scalar function $Q_{-1}(\kappa)$ in (1.17) by $S_{-1}(\kappa)$ given in (3.3b), to obtain that

$$S_1(\kappa) \equiv \sum_{j=0}^{q} c_{j,-1}\kappa^j \neq 0 \;,\; 0 < |\kappa| \leq 1, \tag{3.27}$$

is a sufficient condition for the solvability of the boundary-scheme (3.25). In particular, solvability follows for explicit boundary conditions, where $S_{-1}(\kappa) \equiv$ constant.

Before continuing we want to associate with the boundary conditions (3.1) two concepts - the von Neumann condition and dissipativity - concepts which were previously associated with the basic scheme (1.8). We will find it quite attractive to express our forthcoming stability criteria in terms of these *well-understood* and *easily checkable* concepts.

The boundary-scheme (3.25) has the associated characteristic equation

$$R_0(z,\kappa) \equiv \sum_{j=0}^{q} c_j(z)\kappa^j = 0, \tag{3.28}$$

by means of which, satisfying the von Neumann condition and dissipativity make sense. Upon linking these properties with the boundary-conditions we get:

The boundary conditions (3.1) are said to satisfy the *von Neumann condition* if

$$R_0(z,\kappa{=}e^{i\xi}) \equiv \sum_{j=0}^{q} c_j(z)e^{ij\xi} \neq 0 \;,\; |z| > 1,\; 0 \leq |\xi| \leq \pi, \tag{3.29}$$

and are said to be of *dissipative type* if

$$R_0(z,\kappa{=}e^{i\xi}) \equiv \sum_{j=0}^{q} c_j(z)e^{ij\xi} \neq 0 \;,\; |z| \geq 1,\; 0 < |\xi| \leq \pi. \tag{3.30}$$

We recall that Lemma 2.1 is valid for *any* solvable scheme. Applying that lemma for the *solvable* boundary scheme (3.25), yields the following result.

LEMMA 3.3

For the boundary conditions (3.1) which satisfy the von Neumann condition, we have for z, $|z| \geq 1$,

$$R_0(z,\kappa) \equiv \sum_{j=0}^{q} c_j(z)\kappa^j \neq 0 \ , \ 0 < |\kappa| \leq 1. \tag{3.31}$$

PROOF Take an arbitrary z, $|z| > 1$ and consider the (polynomial) characteristic equation (3.28). By Lemma 2.1I all of its non-zero solutions $\kappa = \kappa(z)$ satisfying $|\kappa(z)| > 1$. Thus there are no non-zero solutions of (3.28) in the closed unit disc, i.e., (3.31) holds.

REMARK 3.4. In the course of proving Lemma 2.1 we used Assumption I, according to which $a_{-r}(z) \neq 0$, $|z| \geq 1$, r denoting the number of left spatial mesh points that the basic scheme rests on. As explained in Remark 2.1, this condition is required in order to assure that multiplying the characteristic equation by factor κ^r does not yield additional zero solutions $\kappa = 0$. We note that upon applying Lemma 2.1 for the *right-sided* boundary scheme (3.25) as done in Lemma 3.3, we are free from requiring, analogously to Assumption I, that $c_0(z) \neq 0$, $|z| \geq 1$, since the index r is vanished in this case, i.e., $\kappa^r = 1$.

By continuity arguments, Lemma 3.3 implies the following immediate result.

COROLLARY 3.3.

For the boundary conditions (3.1) which satisfy the von Neumann condition, we have for z, $|z| \geq 1$

$$R_0(z,\kappa) \equiv \sum_{j=0}^{q} c_j(z)\kappa^j \neq 0 \ , \ 0 < |\kappa| < 1. \tag{3.32}$$

PROOF. By Lemma 3.3, the solutions, $\kappa_j = \kappa_j(z)$, of the characteristic equation (3.28) satisfy for z, $|z| > 1$, $|\kappa_j(z)| > 1$.
Hence, for z, $|z| \geq 1$, these continuous solutions satisfy $|\kappa_j(z)| \geq 1$, and the result follows.

Combining the last corollary with Theorem 3.2, we obtain the following scheme-independent stability criterion of the desired type.

THEOREM 3.3.

The basic scheme (1.8) of dissipative type, together with the boundary conditions (3.1) which satisfy the von Neumann condition, constitute a stable approximation to the outflow problem (1.7).

PROOF. Since the boundary conditions (3.1) satisfy the von Neumann condition, then by Corollary 3.3 we have (3.32) which by the second part of Theorem 3.2 is sufficient for stability.

The last theorem provides a scheme-independent stability criterion for difference approximations whose basic scheme is limited to be of dissipative type. We turn now to the general case, of basic schemes which are not necessarily dissipative. In particular, we refer to the case of non-dissipative schemes where unlike the dissipative case *all* z lying on the unit circle may serve as *generalized* eigenvalues.

THEOREM 3.4.

The basic scheme (1.8) together with the boundary conditions (3.1) of dissipative type, constitute a stable approximation to the outflow problem (1.7).

PROOF Take an arbitrary z with $|z| \geq 1$ and let $\kappa_\ell(z)$, $0 < |\kappa_\ell(z)| \leq 1$, be *any* corresponding inner solution, so that

$$(3.33) \qquad P(z,\kappa_\ell) \equiv \sum_{j=-r}^{p} a_j(z)\kappa_\ell^j = 0.$$

In order to assure stability, it suffices, according to Theorem 3.1, to show that

$$(3.34) \qquad R_0(z,\kappa_\ell) \equiv \sum_{j=0}^{q} c_j(z)\kappa_\ell^j \neq 0.$$

We first note, that since the boundary conditions were assumed to be of dissipative type, they particularly satisfy the von Neumann condition; so Lemma 3.3 and Corollary 3.3 may apply to our case.

Now, for z-values outside the unit disc, $|z| > 1$, (3.34) follows from Lemma 3.3, and for an inner solution which is inside the unit circle, $0 < |\kappa_\ell| < 1$, (3.34) follows from Corollary 3.3.

Therefore, it remains to verify (3.34) for the case that both z and κ_ℓ are lying on the unit circle, i.e.,

$$(3.35) \qquad z = e^{i\varphi}, \ 0 \leq |\varphi| \leq \pi \quad ; \quad \kappa_\ell = e^{i\xi}, \ 0 \leq |\xi| \leq \pi.$$

For an inner solution of the form $\kappa_\ell = e^{i\xi}$, $\xi \neq 0$, (3.34) follows from the dissipativity of the boundary conditions (see (3.30)). So let us consider an inner solution of the form $\kappa_\ell = e^{i\xi}\big|_{\xi=0} = 1$.

By Lemma 3.1, upon approximating the *outflow* problem (1.7), $\kappa_\ell = 1$ is excluded as an *inner* solution corresponding to $z = 1$; and for the remaining z-values, $z = e^{i\varphi}$, $0 < |\varphi| \le \pi$, which may be taken into consideration, we have by (3.33) and by Assumption V

$$|R_0(z=e^{i\varphi},\kappa_\ell=1)| = \Delta(z=e^{i\varphi}) \neq 0 \ , \ 0 < |\varphi| \le \pi \ . \tag{3.36}$$

That is, (3.34) is valid also in the remaining case, and stability follows.

Combining Theorem 3.3 and 3.4 we immediately obtain the following summary result.

COROLLARY 3.4.

Consider the basic scheme (1.8) together with the boundary conditions (3.1) which satisfy the von Neumann condition, as an approximation to the outflow problem (1.7). If either the basic scheme (1.8) or the boundary scheme (3.25) is dissipative, then the approximation is stable.

We note that the stability properties of the boundary scheme, namely, dissipativity and the von Neumann condition, are often *known in advance.* Thus, in applying the last scheme-independent stability criteria summarized in Corollary 3.4, then beside the four basic assumptions (Assumptions I-IV) which the approximation is always assumed to satisfy (see Remark 3.1), it remains to verify the validity of the additional assumptions, Assumptions V and VI.

For the purpose of assuring these assumptions, one may use Lemma 3.2 and Remark 3.3 which imply in particular that Assumption V is automatically fulfilled in the case of two-level boundary conditions, and that the solvability assumption VI is automatically fulfilled in the case of explicit ones.

Yet, referring to the general multi-level implicit case, then Assumptions V and VI are indeed necessary for the validity of the scheme-independent stability criteria given in Theorems 3.3 and 3.4.

Concerning the first of these two, then by Lemma 3.2 it follows that Assumption V is automatically fulfilled in the two-level case. When we raise the question whether stability is maintained also in the general multi-level case involving more than two time steps, the following example shows that the answer to that question is negative. That is, Assumption V is indeed a necessary one.

Consider the non-dissipative Leap-Frog scheme

$$(3.37a) \qquad v_\nu(t+\Delta t) = v_\nu(t-\Delta t) + \lambda a\cdot[v_{\nu+1}(t)-v_{\nu-1}(t)] \ , \ \nu = 1,2,\ldots,$$

together with the solvable consistent boundary condition

$$(3.37b) \qquad v_0(t+\Delta t) = v_0(t-\Delta t) + 2\lambda a\cdot[v_1(t-\Delta t)-v_0(t-\Delta t)].$$

The boundary function associated with (3.37b) is given by

$$(3.38) \qquad R_0(z,\kappa) = 1 - z^{-2}\cdot[1+2\lambda a(\kappa-1)] \ ,$$

and its z-roots, $z = z(\kappa)$, satisfy for $0 < \lambda a \leq 0.5$

$$(3.39) \qquad |z^2(\kappa=e^{i\xi})|^2 < (|1-2\lambda a| + |2\lambda a|)^2 = 1 \ , \ 0 < |\xi| \leq \pi$$

Thus, the explicit (and hence solvable) boundary condition (3.37b) is of dissipative type; so approximation (3.37) fulfills the requirements of both Theorem 3.4 and Assumption VI.

At the same time, the approximation (3.37) is unstable. Indeed the characteristic equation associated with (3.37a)

$$P(z,\kappa) = 1-z^{-2}-z^{-1}\cdot\lambda a(\kappa-\kappa^{-1}) = 0,$$

has exactly one inner root-function, $\kappa = \kappa(z)$, satisfying $\kappa(z=-1) = 1$ (see [6, Lemma 6.2]), and by inserting it into the boundary-function (3.38), we get

$$R_0(z=-1,\kappa=1) = 1-z^{-2}\cdot[1+2\lambda a(\kappa-1)]\Big|_{\substack{z=-1\\ \kappa=1}} = 0. \tag{3.40}$$

Thus, the approximation (3.37), is unstable since it violates the necessary stability condition (3.4) at the point $z = -1$.

The instability of approximation (3.37), despite that it fulfills the requirements of both Theorem 3.4 and Assumption VI, is explained by its failure to satisfy also Assumption V as follows from (3.38):

$$\Delta(z=e^{i\varphi}\big|_{\varphi=-\pi}) = |R_0(z=-1,\kappa=1)| = 0. \tag{3.41}$$

We remark that according to Lemma 3.2, the only possibility of the *three-level* approximation (3.37) to violate Assumption V is at the single point $z = -1$, as we have indeed found in (3.41).

Concerning the solvability Assumption VI, its necessity can be shown by considering *any* two-level 3-point dissipative basic scheme together with zero order accurate boundary condition of the form

$$v_0(t+\Delta t) - \beta v_1(t+\Delta t) = v_0(t) - \beta v_1(t) \ , \ \beta > 1. \tag{3.42}$$

The boundary-function associated with (3.42) which is given by

$$R_0(z,\kappa) = (1-z^{-1})\cdot(1-\beta\kappa) \ , \tag{3.43}$$

satisfies $R_0(z,\kappa=e^{i\xi}) \neq 0$, $|z| > 1$, $0 \leq |\xi| \leq \pi$, hence by (3.29) the boundary condition (3.42) satisfies the von Neumann condition and the entire approximation fulfills the requirements of Theorem 3.3. Furthermore, since the basic scheme was assumed to be two-leveled, then by Lemma 3.2, Assumption V is fulfilled as well. Yet, the approximation is unstable since the boundary-function (3.43) vanishes at $z = 1$ independently of κ-values; hence the necessary stability condition (3.4) is violated. This instability is explained by the failure of the boundary scheme associated with (3.42) to be solvable. Indeed, recalling the solvability definition in (3.26), then by taking $\Psi \equiv 0$ in (3.26a) we find that the grid function $w \equiv \left\{\beta^{-\nu} w_0\right\}_{\nu=-r+1}^{\infty} \in \ell_2(x)$ with arbitrary w_0, satisfies $S_{-1} w_\nu \equiv w_\nu - \beta w_{\nu+1} = 0$, $\nu \geq -r+1$. Thus, we have neither the uniqueness nor the boundedness which is required in (3.26b).

Our study of the outflow problem is completed, and we turn now to discuss some remarks concerning the inflow one.

We first note that all the results which were discussed in previous sections go over unchanged except for Lemma 3.1. The result of this lemma discussing the behavior of inner solutions in the neighbourhood of $z = 1$, depends on whether we approximate the outflow problem or the inflow one.

In the outflow case, Lemma 3.1 is used to exclude the possibility of $\kappa_\ell = 1$ to serve as an *inner* solution corresponding to $z = 1$. In the inflow case, however, the situation is just the contrary; that is, according to Lemma 3.1, $z = 1$ has *always* exactly one corresponding inner solution $\kappa_\ell = 1$.

Now, we recall that all our stability criteria, particularly the scheme-independent ones, were obtained by applying Theorem 3.1 which characterizes stability by requiring that for every z with $|z| \geq 1$ and every corresponding inner solution $\kappa_\ell = \kappa_\ell(z)$, $1 \leq \ell \leq n$, we have $R_0(z,\kappa_\ell) \neq 0$.

We therefore conclude that when dealing with the inflow problem, all our previous stability criteria still hold upon making the additional requirement

$$R_0(z=1,\kappa_\ell=1) \neq 0,$$

a requirement which was automatically excluded by Lemma 3.1, in the outflow case.

Thus for example, referring to the summary result in Corollary 3.4, we obtain for the inflow problem

COROLLARY 3.5.

Consider the basic scheme (1.8) together with the boundary conditions (3.1) which satisfy the von Neumann condition, as an approximation to the inflow problem

(1.7). If either the basic scheme (1.8) or the boundary scheme (3.25) is dissipative and if in addition we have

(3.42) $$R_0(z=1,\kappa=1) \neq 0,$$

then the approximation is stable.

We note that when the additional condition (3.42)

$$R_0(z,\kappa)\big|_{z=\kappa=1} = \sum_{j=0}^{q} c_j(z)\big|_{z=1} \neq 0$$

holds, then according to (1.10a) the boundary conditions must be *inconsistent* and in fact have no accuracy with respect to the differential equation. This indeed makes sense since one cannot expect the stable approximation (1.8), (3.1) whose values are uniquely determined in the quarter-plane $x, t \geq 0$, to be consistent with the *inflow* problem (1.7) which is *not* uniquely determined unless extra boundary data is supplied as given in (1.7b). Thus in general, consistent boundary conditions of translatory type approximating the outflow problem are of no value when dealing with the inflow one. Yet, there is one important case which we shall now consider. That is, when the missing boundary values are computed via summations of the form (see [7, Theorem 6])

(3.43) $$\sum_{j=0}^{q} v_{\mu+j}(t) = g_\mu(t), \qquad g_\mu(t)\in \ell_2(t),\ \mu \equiv 0,-1,\ldots,-r+1.$$

For the purpose of determining stability, we consider the boundary-function

associated with the homogeneous part of (3.43), which is given by $R_0(\kappa) = \sum_{j=0}^{q} \kappa^j$.
We have $R_0(\kappa) \neq 0$ for all $\kappa \neq 0$ and hence by Corollary 3.1, the stability of both the outflow and particularly the inflow approximation (1.8), (3.43) is assured. Setting q to be zero, we obtain the well-known result [7], [10] of the unconditional stability of the approximation whose boundary values are determined by arbitrary bounded inhomogeneous terms $v_\mu(t) = g_\mu(t)$, $\mu = 0,-1,\ldots,-r+1$. Indeed, this result was mentioned earlier in Section 1.1, where it was used to assure the unconditional stability of the inflow scalar components of the vector approximation (1.5), (1.3).

4. EXAMPLES OF SCHEME-INDEPENDENT STABILITY INVESTIGATIONS.

In this chapter we study some examples of translatory boundary conditions which together with corresponding basic schemes constitute stable approximations to the *outflow* problem (1.7).

For that purpose, we apply the scheme-independent criteria of the previous chapter, so that stability is not restricted to a *specific basic scheme*. That is, the acquired stability is valid for a *family* of approximations which consists of the boundary conditions together with *any* basic scheme having some general property (the "familial" property) such as dissipativity, two-levelness, etc.

REMARK 4.1. It is of course understood that beside requiring the basic schemes to satisfy some general ("familial") property which follows from the scheme-independent stability analysis, all basic schemes considered must satisfy the four basic assumptions, Assumptions I-IV (see Remark 3.1). In particular, we refer to the stability assumption, Assumption III, which may lead to impose some restriction on the time step used Δt.

We note that the stability criteria given in Theorems 3.3 and 3.4 are independent of the index r, which denotes the number of boundary values to be computed at each time level $v_\mu(t)$, $\mu = 0,-1,\dots,-r+1$. Hence, verifying the stability in the simpler case of computing a single boundary value, $v_0(t)$, (complementing for example, a 3-point basic scheme) requires no more effort than the stability verification in the general translatory case of r boundary values, $r > 1$, complementing a basic scheme of the general form (1.8).

The above observations are particularly relevant to those boundary treatments which are considered below, and whose stability is already discussed in the literature. However, the verification of stability given below has two specific features: first, because the stability investigation is independent of the basic scheme and of the solutions of the corresponding characteristic equation, then the procedure becomes much shorter; secondly, the translatory nature of the boundary treatment assures that the acquired stability is valid for *any* basic stable scheme and not necessarily for 3-point ones.

Let us turn then to the examples themselves. and consider first boundary conditions which complement *any dissipative basic scheme*.

EXAMPLE 4.1. (example (6.11) in [6]). Let the boundary conditions be determined by oblique Lagrangian extrapolation of order m-1:

$$(4.1a) \qquad v_\mu(t+\Delta t) = \sum_{j=1}^{m} \binom{m}{j}(-1)^{j+1} v_{\mu+j}[t-(j-1)\Delta t] \ , \ \mu = 0,-1,\dots,-r+1.$$

The boundary-function associated with (4.1a) is given by

$$(4.1b) \qquad R_0(z,\kappa) = 1-\sum_{j=1}^{m} \binom{m}{j}(-1)^{j+1} z^{-j}\kappa^j \equiv (1-z^{-1}\kappa)^m,$$

and by equating to zero, we get that the z-solutions of (4.1b) satisfy

$$|z(\kappa=e^{i\xi})| = |e^{i\xi}| = 1 \ , \ \forall\xi.$$

Thus, the explicit and hence (by Remark 3.3) *solvable* boundary conditions (4.1a) are of non-dissipative type so they satisfy the *von Neumann condition*. According to (4.1b) we also have $z(\kappa=1) = 1$, so $\Delta(z=e^{i\varphi}) \geq |R_0(z=e^{i\varphi},\kappa=1)|>0$, $0<|\varphi|\leq\pi$, i.e., *Assumption* V is valid and stability follows by Theorem 3.3.

<u>EXAMPLE 4.2</u>. (example (6.3c) in [6], example (3.4) in [14]). Let the boundary conditions be generated by the Box-Scheme, i.e.,

(4.2a) $$v_\mu(t+\Delta t) + v_{\mu+1}(t+\Delta t) - \lambda a\cdot[v_{\mu+1}(t+\Delta t) - v_\mu(t+\Delta t)] =$$

$$= v_\mu(t) + v_{\mu+1}(t) + \lambda a\cdot[v_{\mu+1}(t)-v_\mu(t)] \ , \quad \mu = 0,-1,\ldots,-r+1.$$

The boundary-function associated with (4.2a) is given by

(4.2b) $$R_0(z,\kappa) = 1+\kappa-\lambda a(\kappa-1) - z^{-1}\cdot[1+\kappa+\lambda a(\kappa-1)] \ ,$$

and by equating to zero, we get that its z-solutions satisfy

$$|z(\kappa=e^{i\xi})| = \left|\frac{1+e^{i\xi}+\lambda a(e^{i\xi}-1)}{1+e^{i\xi}-\lambda a(e^{i\xi}-1)}\right| = 1 \ , \ \forall\xi.$$

Thus, boundary conditions (4.2a) are of non-dissipative type so they satisfy the *von Neumann condition*. Approximating the outflow problem (a>0) we have

$$\mathrm{Re}[S_{-1}(\kappa)] = 1+\mathrm{Re}(\kappa) + \lambda a\cdot[1-\mathrm{Re}(\kappa)] \neq 0 \ , \ |\kappa| \leq 1.$$

Hence (see Remark 3.3), the boundary conditions (4.2a) are *solvable* (i.e., *Assumption VI* holds), and since they are also two-leveled then (by Lemma 3.2)

Assumption V holds as well. Therefore we may apply Theorem 3.3 obtaining stability.

In view of the stability discussion in Examples 4.1, 4.2, it follows that *any dissipative basic scheme together with translatory boundary conditions which are generated by oblique extrapolation or by the Box-Scheme, constitute a stable approximation to the outflow problem (1.7).*

EXAMPLE 4.3. ([2] , example (6.2b) in [15]). Let the boundary conditions be generated by the right-sided wieghted Euler scheme, i.e.,

$$v_\mu(t+\Delta t) = v_\mu(t-\Delta t) + 2\lambda a\cdot[v_{\mu+1}(t)-0.5\cdot(v_\mu(t+\Delta t) + v_\mu(t-\Delta t))],$$

$$\mu = 0,-1,\ldots,-r+1.$$

The boundary-function associated with (4.3a) is

$$R_0(z,\kappa) = 1-z^{-2}-2\lambda a\cdot[\kappa\cdot z^{-1}-0.5\cdot(1+z^{-2})], \tag{4.3b}$$

and by equating to zero, we find that its z-solutions are given by

$$z(\kappa=e^{i\xi}) = e^{i\xi}\,\frac{\lambda a\pm b(\xi)}{\lambda a+1}\,, \quad b(\xi) = \sqrt{(\lambda a)^2+e^{-2i\xi}[1-(\lambda a)^2]}. \tag{4.3c}$$

In order to assure stability, we restrict the time-step Δt by requiring the Courant-Friedrichs-Levi (CFL) condition

(4.3d) $$0 < \lambda a \leq 1.$$

We note that the CFL requirement (4.3d) is automatically fulfilled in the case of an explicit basic scheme, since by our Assumption III, the basic scheme must satisfy the von Neumann condition.

Having (4.3d), it follows that $|b(\xi)| \leq 1$, $0 \leq |\xi| \leq \pi$. Thus, the explicit and hence (by Remark 3.3) *solvable* boundary conditions (4.3a) satisfy the *von Neumann condition*

$$|z(\kappa=e^{i\xi})| \leq \frac{\lambda a+|b(\xi)|}{\lambda a+1} \leq 1 , \quad \forall\xi.$$

According to (4.3c) we have $z(\kappa=1) = \frac{\lambda a\pm 1}{\lambda a+1} \neq e^{i\varphi}$, $0 < |\varphi| \leq \pi$, so $\Delta(z=e^{i\varphi}) \geq |R_0(z=e^{i\varphi},\kappa=1)| > 0$, $0 < |\varphi| \leq \pi$, i.e., *Assumption V* is valid. We may apply now Theorem 3.3 to conclude that *any approximation to the outflow problem (1.7) which satisfies the CFL condition (4.3d)*(1)*, consisting of a dissipative basic scheme together with the translatory boundary conditions (4.3a), is stable.*

We turn now to study general difference approximation consisting of any basic scheme (which is not necessarily of dissipative type) together with the following boundary treatments.

EXAMPLE 4.4. (example (6.3b) in [6] , example (3.2) in [14]). Let the boundary conditions be generated by the right-sided explicit Euler scheme, i.e.,

(4.4a) $$v_\mu(t+\Delta t) = v_\mu(t) + \lambda a\cdot[v_{\mu+1}(t)-v_\mu(t)],\ \mu = 0,-1,\dots,-r+1.$$

(1) A further restriction on the time step Δt may arise from the stability requirement made in Assumption III (see Remark 4.1).

The boundary-function associated with (4.4a) is given by

(4.4b) $$R_0(z,\kappa) = 1-z^{-1}\cdot[1+\lambda a(\kappa-1)],$$

and by equating to zero, we get that its z-solutions satisfy

$$|z(\kappa=e^{i\xi})|^2 = (1-\lambda a+\lambda a\cdot\cos\xi)^2 + (\lambda a\cdot\sin\xi)^2 < (|1-\lambda a|+|\lambda a|)^2 \ , \ 0 < |\xi| \leq \pi.$$

Thus, requiring the CFL condition

(4.4c) $$0 < \lambda a \leq 1,$$

we see that the boundary conditions (4.4a) are of *dissipative* type. Since the boundary conditions (4.4a) are also explicit and two-leveled then by Lemma 3.2 and Remark 3.3, both *Assumptions V and VI* are valid and stability follows from Theorem 3.4.

<u>EXAMPLE 4.5</u>. (example (3.3) in [14]). Let the boundary conditions be generated by the right-sided implicit Euler scheme, i.e.,

(4.5a) $$v_\mu(t+\Delta t) - \lambda a\cdot[v_{\mu+1}(t+\Delta t) - v_\mu(t+\Delta t)] = v_\mu(t) \ , \ \mu = 0,-1,\ldots,-r+1.$$

The boundary-function associated with (4.5a) is given by

(4.5b) $$R_0(z,\kappa) = 1-\lambda a(\kappa-1)-z^{-1},$$

and by equating to zero, we get that its z-solutions satisfy in the outflow case $(a>0)$

$$|z(\kappa=e^{i\xi})|^2 = [(1+\lambda a-\lambda a\cdot\cos\xi)^2+(\lambda a\cdot\sin\xi)^2]^{-1} < [|1+\lambda a|-|\lambda a|]^{-1} = 1,\ 0 < |\xi| \leq \pi.$$

Thus, boundary conditions (4.5a) are of *dissipative* type. Approximating the outflow problem $(a>0)$ we have

$$\mathrm{Re}[S_{-1}(\kappa)] = 1+\lambda a\cdot[1-\mathrm{Re}(\kappa)] \neq 0\ ,\ |\kappa| \leq 1.$$

Hence (see Remark 3.3), the boundary conditions (4.5a) are *solvable* (i.e., *Assumption VI* holds), and since they are also two-leveled then (by Lemma 3.2) *Assumption V* holds as well. Therefore we may apply Theorem 3.4 obtaining stability.

In view of the stability discussion in Examples 4.4 and 4.5, we may conclude that *if the boundary conditions (3.1) are generated by a stable right-sided explicit Euler scheme or by the right-sided implicit Euler scheme, then the entire approximation (1.8), (3.1) is stable.*

EXAMPLE 4.6. Let the boundary conditions be of the form

$$(4.6a)\quad (1+\lambda a)v_\mu(t+\Delta t) + (1-\lambda a)v_{\mu+1}(t+\Delta t) =$$

$$= 2\lambda a\cdot[v_{\mu+1}(t)-v_\mu(t)] + (1-\lambda a)v_\mu(t-\Delta t) + (1+\lambda a)v_{\mu+1}(t-\Delta t),\ \mu = 0,-1,\dots,-r+1.$$

The boundary-function associated with (4.6a) is

$$(4.6b) \qquad R_0(z,\kappa) = 1+\lambda a+(1-\lambda a)\kappa-z^{-1}\cdot 2\lambda a(\kappa-1)-z^{-2}\cdot[(1-\lambda a) + (1+\lambda a)\kappa],$$

and by equating to zero, we find that its z-solutions are given by

$$(4.6c) \qquad z(\kappa=e^{i\xi}) = \frac{\lambda ai\cdot\sin\xi/2 \pm \cos\xi/2}{-\lambda ai\cdot\sin\xi/2 + \cos\xi/2},$$

and hence $|z(\kappa=e^{i\xi})| = 1$. Thus, the boundary conditions (4.6a) are of non-dissipative type so they satisfy the *von Neumann condition*. Furthermore, we have in the outflow case (a>0)

$$\mathrm{Re}[S_{-1}(\kappa)] = 1+\mathrm{Re}(\kappa) + \lambda a\cdot[1-\mathrm{Re}(\kappa)] \neq 0 \ , \ |\kappa| \leq 1,$$

and hence (see Remark 3.3) boundary conditions are also ***solvable***, i.e., *Assumption VI* holds. To assure stability via Theorem 3.3. it then remains to verify Assumption V.

Now, since by (4.6c) we have $z(\kappa=1) = \pm 1$, hence

$$\Lambda(z=e^{i\varphi}) \geq |R_0(z=e^{i\varphi},\kappa=1)| > 0 \ , \ 0 < |\varphi| < \pi \ ,$$

and therefore the validity of Assumption V follows upon requiring

$$\Delta(z=-1) = |P(z=-1,\kappa=1)| \neq 0.$$

Here, $P(z,\kappa)$ denotes as usual, the characteristic function associated with the basic scheme.

Applying Theorem 3.3 we conclude that *the boundary conditions (4.6a), which complement any dissipative basic scheme whose characteristic function $P(z,\kappa)$ satisfies*

$$P(z=-1,\kappa=1) \tag{4.6d}$$

constitute a stable approximation to the outflow problem (1.7).

Since by Lemma 2.2I condition (4.6d) which may be rewritten in the form

$$\sum_{j=-r}^{p} a_j(z=-1) \neq 0$$

is automatically fulfilled in the two-level case, we obtain that stability follows whenever the boundary conditions (4.6a) complement *any two-level* dissipative basic scheme. We note, however, that in the general case of multi-level basic schemes involving more than two time-steps, the additional requirement (4.6d) is indeed a necessary one as shown by the following example:

Consider the three-leveled dissipative scheme (see (3.24))

$$v_\nu(t+\Delta t) = [I - \frac{\epsilon}{16}(E-I)^2(I-E^{-1})^2]v_\nu(t-\Delta t) + \lambda a(E-E^{-1})v_\nu(t), \tag{4.7a}$$

$$\nu = 1,2,\ldots,\ \lambda a \leq 1-\epsilon,\ \epsilon < 1,$$

and let the boundary value $v_0(t+\Delta t), v_{-1}(t+\Delta t)$ be computed by (4.6a), i.e., (4.7) $\quad (1+\lambda a)v_{\mu}(t+\Delta t) + (1-\lambda a)v_{\mu+1}(t+\Delta t) =$

$$= 2\lambda a\cdot[v_{\mu+1}(t)-v_{\mu}(t)] + (1-\lambda a)v_{\mu}(t-\Delta t) + (1+\lambda a)v_{\mu+1}(t-\Delta t), \quad \mu = 0,-1.$$

The characteristic equation associated with the basic scheme (4.7a), has exactly one inner root function $\kappa = \kappa(z)$ satisfying $\kappa(z=-1) = 1$, and by inserting into the associated boundary-function we get

$$R_0(z=-1\,, \kappa=1) = 1+\lambda a-(1-\lambda a)\kappa-z^{-1}\cdot 2\lambda a(\kappa-1)-z^{-2}\cdot[(1-\lambda a)+(1+\lambda a)\kappa]\Big|_{\substack{z=-1\\ \kappa=1}} = 0.$$

Thus, approximation (4.7) is unstable due to the violation of the necessary stability condition (3.4) at the point $z = -1$.

Recalling the summary result which follows Example 4.6, we find that approximation (4.7) satisfies all the required hypothesis except for condition (4.6d). Indeed, we have

$$\text{(4.7c)} \qquad P(z=-1,\kappa=1) = 1-z^{-1}\cdot\lambda a(\kappa-\kappa^{-1})-z^{2}\cdot[1-\frac{\epsilon}{16}(\kappa-1)^{2}\cdot(1-\kappa^{-1})^{2}]\Big|_{\substack{z=-1\\ \kappa=1}} = 0.$$

We remark that according to Lemma 3.2, $z = -1$ is the only possible point for the *three-level* basic scheme (4.7a) to violate Assumption V, as we have indeed found in (4.7c).

We close this chapter by considering difference approximations to the two space dimensional problem

$$\partial u(x,y,t)/\partial t = a\partial u(x,y,t)/\partial x + b\partial u(x,y,t)/\partial y \ , \ a > 0 \ ; \ u(x,y,0) = f(x,y),$$

in the quarter space $x \geq 0$, $t \geq 0$, $-\infty < y < \infty$. The analysis of such initial-boundary problems in both the differential case (see [9]) and the difference case (see for example [1]) can be carried out by Fourier transforming with respect to the variable y; thus obtaining a one space dimensional problem of the type analyzed in the previous chapters. To be more precise, let Δx, Δy be the spatial mesh width such that $\lambda_x \equiv \Delta t/\Delta x$ = constant, $\lambda_y \equiv \Delta t/\Delta y$ = constant and denote by $v_{\nu,\zeta}(t) \equiv v(\nu\Delta x,\zeta\Delta y,t)$ the approximated grid function. Then, Fourier transforming in the y-direction (with dual variable η) and Fourier-Laplace transforming with respect to the time-variable t leads one to search for normal modes of the form $v_{\nu,\zeta}(t) = z^n\kappa^j e^{i\zeta\eta}$. Upon substituting such modes as a grid solution for a given difference scheme, we obtain the corresponding characteristic equation. If in particular the scheme is the one generates the translatory boundary conditions considered, we obtain the associated boundary-function which determines the stability properties of these conditions. Both the characteristic and boundary functions involved in the two space dimensional case are dependent on the extra parameter η, and our former results are still valid in this case since all estimates made are uniform in η (see [6],[8]). The only exception is that of Lemma 3.1, according to which the possibility of $\kappa = 1$ to serve as an inner solution corresponding to $z = 1$, is excluded in the outflow case ($a>0$). The proof of Lemma 3.1 is based on the consistency condition, so its validity in the two space dimensional case is restricted to the single point $\eta = 0$. Therefore, for the result of

Lemma 3.1 to be valid *independently of the extra parameter* η, the additional requirement

(4.8) $$R_0(z=1,\kappa=1,\eta) \neq 0 \ , \quad 0 < |\eta| \leq \pi,$$

must be fulfilled. Thus, to apply our scheme-independent stability criteria for a two space dimensional approximation, we first determine the boundary stability properties by employing the associated boundary-function. Then it remains to check whether the approximation meets the additional assumptions V and VI, and whether condition (4.8) is fulfilled.

We note that when verifying the validity of Assumption V in the two space dimensional case, one may no longer use Lemma 3.2 in which conditions for the validity of Assumption V for two- and three-leveled schemes, are discussed. Indeed, the lemma follows from Lemma 2.2 whose proof is based on the zero order accuracy condition, i.e., when dealing with the boundary conditions we have $R_0(z=1,\kappa=1,\eta=0) = 0$. Thus, the conclusions of Lemma 2.2 and Lemma 3.2 hold only in the *neighbourhood* of $\eta = 0$ and are not necessarily valid for all η, $0 \leq |\eta| \leq \pi$.

EXAMPLE 4.7. Let the boundary conditions be generated by the right-sided explicit Euler scheme, i.e.,

(4.9a)
$$v_{\mu,\zeta}(t+\Delta t) = v_{\mu,\zeta}(t) + \lambda_x a\cdot[v_{\mu+1,\zeta}(t)-v_{\mu,\zeta}(t)] + \lambda_y b\cdot[v_{\mu,\zeta+1}(t)-v_{\mu,\zeta}(t)] \ ,$$
$$\mu = 0,-1,\dots,-r+1, \ -\infty < \zeta < \infty.$$

The boundary-function associated with (4.9a) is given by

(4.9b) $$R_0(z,\kappa,\eta) = 1 - z^{-1}\cdot[1+\lambda_x a\cdot(\kappa-1)+\lambda_y b\cdot(e^{i\eta}-1)],$$

and by equating to zero we get that its z-solutions satisfy

(4.9c) $$|z(\kappa=e^{i\xi},\eta)|^2=(1-\lambda_x a-\lambda_y b+\lambda_x a\cdot\cos\xi+\lambda_y b\cdot\cos\eta)^2 + (\lambda_x a\cdot\sin\xi+\lambda_y b\cdot\sin\eta)^2 < \\ < (|1-\lambda_x a-\lambda_y b| + |\lambda_x a+\lambda_y b|)^2 , \quad 0 < |\xi| \le \pi, \quad 0 \le |\eta| \le \pi.$$

Then, upon imposing the CFL condition

(4.9d) $$0 < \lambda_x a + \lambda_y b \le 1$$

we see that the explicit and hence (by Remark 3.3) *solvable* boundary conditions (4.9a) are of *dissipative* type. By (4.9d) it follows that in the outflow case (a>0) we have $\lambda_y b \le 1$, hence

(4.9e) $$|z(\kappa=1,\eta)|^2 = (1-\lambda_y b+\lambda_y b\cdot\cos\eta)^2 + (\lambda_y b\cdot\sin\eta)^2 <|1-\lambda_y b|+|\lambda_y b|= 1, \\ 0 < |\eta| \le \pi,$$

so condition (4.8) is fulfilled. Consistency implies that $z(\kappa=1,\eta=0) = 1$ (see (4.9e)) and together with (4.9e) we finally get

$$\Delta(z=e^{i\varphi},\eta) \geq |R_0(z=e^{i\varphi},\kappa=1,\eta)| > 0 \ , \ 0 < |\varphi| \leq \pi, \ 0 \leq |\eta| \leq \pi,$$

i.e., *Assumption V* holds. Therefore we may apply Theorem 3.4 concluding that *if the CFL condition (4.9d) is fulfilled*[1]*, then the outflow boundary conditions (4.9a) always maintain stability independently of the interior scheme.*
Boundary conditions (4.9a) are generated by an obvious extension of the usual one space dimensional right-sided Euler scheme discussed in Example 4.4. Another possible extension is given by (see example (2.5) in [1])

$$v_{\mu,\zeta}(t+\Delta t) = v_{\mu,\zeta}(t) + \lambda_x a\cdot[v_{\mu+1,\zeta}(t)-v_{\mu,\zeta}(t)] + 0.5\cdot\lambda_y b\cdot[v_{\mu,\zeta+1}(t)-v_{\mu,\zeta-1}(t)],$$

(4.10a)

$$\mu = 0,-1,\ldots,-r+1, \quad -\infty < \zeta < \infty.$$

These boundary conditions are unstable in the sense that the z-solutions of the associated boundary-function

(4.10b) $$R_0(z,\kappa,\eta) = 1-z^{-1}\cdot[\lambda_x a(\kappa-1) + i\lambda_y b\cdot\sin\eta],$$

satisfy

(4.10c) $$|z(\kappa=e^{i\xi},\eta)|^2 = (1-\lambda_x a+\lambda_x a\cdot\cos\xi)^2 + (\lambda_x a\cdot\sin\xi + \lambda_y b\cdot\sin\eta)^2,$$

hence $|z(\kappa=e^{i\xi},\eta)_{|\xi=0}|^2 = 1 + (\lambda_y b\cdot\sin\eta)^2 > 1, \ 0 < |\eta| \leq \pi$. Thus, boundary

(1) A further restriction on the time step Δt may arise from the stability requirement made in Assumption III (see Remark 4.1).

conditions (4.10a) fail to satisfy the von Neumann condition for *all* η, $0\leq|\eta|\leq\pi$ and our scheme-independent stability criteria are inapplicable in this case. The question of stability is, in this case, dependent on the basic scheme utilized.

A further possibility to extend the one space dimensional Euler scheme (4.4a) is considered in the following example.

EXAMPLE 4.8. (example (2.6) in [1]). Let the boundary conditions be of the form

$$v_{\mu,\zeta}(t+\Delta t) = v_{\mu,\zeta}(t) + 0.5\cdot\lambda_x a\cdot[(v_{\mu+1,\zeta+1}(t)+v_{\mu+1,\zeta}(t))-(v_{\mu,\zeta+1}(t)+v_{\mu,\zeta}(t)] +$$

(4.11a)

$$+ 0.5\cdot\lambda_y b\cdot[(v_{\mu+1,\zeta+1}(t)+v_{\mu,\zeta+1}(t))-(v_{\mu+1,\zeta}(t)+v_{\mu,\zeta}(t))],$$

$$\mu = 0,-1,\ldots,-r+1,\ -\infty < \zeta < \infty.$$

To simplify the computations we shall consider the case where $\lambda_x a = \lambda_y b$. The boundary-function associated with (4.11a) is given by

(4.11b) $$R_0(z,\kappa,\eta) = 1-z^{-1}\cdot\lambda_x a\cdot(\kappa e^{i\eta}-1)$$

and by equating to zero we get that its z-solutions satisfy

(4.11c) $$|z(\kappa=e^{i\xi},\eta)|^2 = [1-\lambda_x a+\lambda_x a\cdot\cos(\xi+\eta)]^2 + [\lambda_x a\cdot\sin(\xi+\eta)]^2 \leq$$

$$\leq (|1-\lambda_x a| + |\lambda_x a|)^2,\ 0\leq|\xi|,|\eta|\leq\pi .$$

Then, upon imposing the CFL condition

(4.11d) $$0 < \lambda_x a = \lambda_y b \leq 1$$

we see that the explicit and hence (by Remark 3.3) *solvable* boundary conditions (4.11a) satisfy the *von Neumann condition*. By (4.11c) and (4.11d) we have

$$|z(\kappa=1,\eta)|^2 = (1-\lambda_x a+\lambda_x a\cdot\cos\eta)^2 + (\lambda_x a\cdot\sin\eta)^2 < (|1-\lambda_x a| + |\lambda_x a|)^2 = 1,$$

(4.11e)

$$0 < |\eta| \leq \pi,$$

and hence condition (4.8) is fulfilled. Consistency implies that $z(\kappa=1,\eta=0) = 1$ (see (4.11e)) and together with (4.11e) we finally get

$$\Delta(z=e^{i\varphi},\eta) \geq |R_0(z=e^{i\varphi},\kappa=1,\eta)| > 0 \ , \ 0 < |\varphi| \leq \pi, \ 0 \leq |\eta| \leq \pi,$$

i.e., *Assumption V* holds. Therefore we may apply Theorem 3.3 concluding that ***if the CFL condition (4.11d) is fulfilled,***(1) ***then the outflow boundary conditions (4.11a) in conjunction with any dissipative basic scheme constitute together a stable approximation.***

In the manner of the last two examples, one may consider various two space dimensional boundary treatments which extend the corresponding one space dimensional ones. We choose to consider an example which is based on extrapolation.

EXAMPLE 4.9. (example (2.4) in [1]). Let the boundary conditions be determined

(1) A further restriction on the time step Δt may arise from the stability requirement made in Assumption III (see Remark 4.1).

by oblique extrapolation *along the characteristic plane* (see Example 4.1), i.e.,

$$(4.12a)\quad v_{\mu,\zeta}(t+\Delta t) = \sum_{j=1}^{m} \binom{m}{j}(-1)^{j+1} v_{\mu+j,\zeta+j}[t-(j-1)\Delta t],\ \mu=0,-1,\dots,-r+1,-\infty<\zeta<\infty.$$

The boundary-function associated with (4.12a) is given by

$$(4.12b)\qquad R_0(z,\kappa,\eta) = 1 - \sum_{j=1}^{m} \binom{m}{j}(-1)^{j+1} z^{-j}\kappa^j e^{ij\xi} \equiv (1-z^{-1}\kappa e^{i\eta})^m ,$$

and by equating to zero we get that its z-solutions satisfy

$$(4.12c)\qquad |z(\kappa=e^{i\xi},\eta)| = |e^{i\xi}e^{i\eta}| = 1,\quad 0 \le |\xi|,\ |\eta| \le \pi .$$

Thus, the explicit and hence (by Remark 3.3) ***solvable*** boundary conditions (4.12a) are of non-dissipative type so they satisfy the *von Neumann condition*. In addition we have

$$R_0(z=1,\kappa=1,\eta) = (1-e^{i\eta})^m \neq 0 \ ,\ 0 < |\eta| \le \pi,$$

i.e., condition (4.8) is fulfilled. Thus to assure stability via Theorem 3.3, it then remains to verify Assumption V. Since by (4.12b) we have

$$R_0(z=e^{i\varphi},\kappa=1,\eta)_{|\varphi=\eta} = (1-e^{-i\varphi}e^{i\eta})^m_{|\varphi=\eta} = 0,$$

the assumption is reduced in this case to the requirement

(4.12d) $\Delta(z=e^{i\varphi},\eta) = |P(z=e^{i\varphi},\kappa=1,\eta)| > 0, \; 0 < |\varphi| \leq \pi, \; 0 < |\eta| \leq \pi,$

$P(z,\kappa,\eta)$ denoting as usual the corresponding characteristic function associated with the basic scheme.

Applying Theorem 3.3 we conclude that ***the extrapolatory outflow boundary conditions (4.12a), when complementing any dissipative basic scheme whose characteristic function*** $P(z,\kappa,\eta)$ ***satisfies*** $P(z=e^{i\varphi},\kappa=1,\eta) \neq 0$, $0<|\varphi|<\pi$, $0<|\eta|\leq\pi$, ***constitute a stable approximation.***

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